A New Derivation of Lorentz Transformation.

As remarked by Levy-Leblond,[1] very little freedom is allowed for the choice of a relativity group, so that the Poincaré group is an almost unique solution to the problem[2]. In his original paper, Einstein derived the Lorentz transformation from the (sometimes implicit) successive assumptions of (i) linearity, (ii) invariance of c, the light velocity in vacuum, (iii) existence of a composition law, (iv) existence of a neutral element, and (v) reflection invariance.

But one may demonstrate that the postulate of the invariance of some absolute velocity is not necessary for the construction of the special theory of relativity. Indeed it was shown by Levy-Leblond[1] that the Lorentz transformation may be obtaines through six *sucessive* constraints: {1} homogeneity of space-time (translated as the linearity of the transformation of coordinates), {2} isotropy of space-time (translated as reflection invariance), {3} group structure (i.e. {3.1} existence of a neutral element, {3.2} of an inverse transformation, and {3.3} of a composition law yielding a new transformation which is a member of the group, viz, which is internal), and {4} the causality condition. The last group axiom, associativity, is in fact straightforward in this case and leads to no new constraint.

Actually this set of hypotheses is still overdetermined to derive the Lorentz transformation. We shall indeed demonstrate hereafter that the Lorentz transformation may be obtained from *only the assumptions of* {a} *linearity*, {b} *internal composition law*, and {c} *reflection invariance*. All the other assumptions, in particular the postulate of the existence of an inverse tansformation which is a member of the group, may be *derived as consequences* of these purely mathematical constraints. The importance of this result, especially concerning scale relativity, is that we do not have to postulate a full group law in order to get the Lorentz behaviour: the hypothesis of a semigroup structure is sufficient.

Let us start from a linear transformation of coordinates: (Voir la démonstration de ces équations par Jean-Marc Lévy-Leblond, équations (8a) et(8b), qui se sert de l'homogénéïté de l'espace-temps pour les obtenir. Pour Laurent Nottale, ces équations apparaissent sans hypothèse puisqu'elles correspondent au choix le plus simple : la linéarité.)

$$x' = a(v)x - b(v)t, \tag{1a}$$
$$t' = \alpha(v)t - \beta(v)x. \tag{1b}$$

In these equations and in the whole section, the coordinates x and t do not denote *a priori* lenghts and times, but may refer to any kind of variables having the mathematical proprieties considered. Equation (1) may be written as $x' = a(v)[x - (b/a)t]$. But we may *define* the "velocity" v as $v = b/a$ (Voir la démonstration de Jean-Marc Lévy-Leblond, équation (9), pour l'identification de v avec b/a) (in case of motion laws, this is indeed the velocity in the usual meaning; in the case of scale laws, this is the state of scale or "scale velocity").

Then, whithout any loss of generality, linearity alone leads to the general form

$$x' = \gamma(v)[x - vt], \tag{2a}$$
$$t' = \gamma(v)[A(v)t - B(v)x], \tag{2b}$$

where $\gamma(v)$ now stands for $a(v)$, and where A and B are new functions of v.
Let us now perform two successive transformations of the form (2):

$$x' = \gamma(u)[x - ut], \tag{3a}$$
$$t' = \gamma(u)[A(u)t - B(u)x], \tag{3b}$$

$$x'' = \gamma(v)[x' - vt'], \tag{3c}$$
$$t'' = \gamma(v)[A(v)t' - B(v)x']. \tag{3d}$$

Ce qui donne :

$$\begin{aligned} x'' &= \gamma(u)\gamma(v)(x - ut) - v[A(u)t - B(u)x] \\ t'' &= \gamma(v)A(v)\gamma(u)[A(u)t - B(u)x] - B(v)\gamma(u)[x - ut] \\ &= \gamma(u)\gamma(v)[1 + B(u)v][x - \frac{u + A(u)v}{1 + B(u)v}t] \end{aligned}$$

This results in the transformation

$$x'' = \gamma(u)\gamma(v)[1 + B(u)v]\left[x - \frac{u + A(u)v}{1 + B(u)v}t\right], \tag{4a}$$
$$t'' = \gamma(u)\gamma(v)[A(u)A(v) + B(v)u]\left[t - \frac{A(v)B(u) + B(v)}{A(u)A(v) + B(v)u}x\right]. \tag{4b}$$

Then the principle of relativity tells us that the composed transformation (4) keeps the same form as the initial one (2), in terms of a composed velocity w given by the factor of t in (4a).
C'est à dire, on doit aussi avoir:

$$x'' = \gamma(w)[x - wt],$$
$$t'' = \gamma(w)[A(w)t - B(w)x].$$

We get four conditions:

$$w = \frac{u + A(u)v}{1 + B(u)v}, \tag{5a}$$
$$\gamma(w) = \gamma(u)\gamma(v)[1 + B(u)v], \tag{5b}$$
$$\gamma(w)A(w) = \gamma(u)\gamma(v)[A(u)A(v) + B(v)u], \tag{5c}$$
$$\frac{B(w)}{A(w)} = \frac{A(v)B(u) + B(v)}{A(u)A(v) + B(v)u}. \tag{5d}$$

Our third postulate is reflection invariance. It reflects the fact that the choice of the orientation of the x (and x') axis is completely arbitrary, and should be indistinguishable from the alternative choice $(-x, -x')$. With this new choice, the transformation (3) becomes

$$-x' = \gamma(u')(-x - u't),$$
$$t' = \gamma(u')[A(u')t + B(u')x]$$

in terms of the value u' taken by the relative velocity in the new orientation. The requirement that the two orientations be indistinguishable yields $u' = -u$. (Voir la démonstration de Jean-Marc Lévy-Leblond, équation (14), qui se sert de l'isotropie de l'espace (reflection invariance)). This leads to parity relations[1] for the three unknown functions γ, A and B:

$$\gamma(-v) = \gamma(v), \quad A(-v) = A(v), \quad B(-v) = -B(v). \tag{6}$$

(et $B(0) = -B(0)$ donc $B(0) = 0$).
Combining Eqs. (5a), (5b) and (5c) yields the relation. Je détaille un peu :
On réécrit l'équation (5c) :

$$A(w) = \frac{\gamma(u)\gamma(v)[A(u)A(v) + B(v)u]}{\gamma(w)}$$

En remplaçant $\gamma(w)$ donné par l'équation (5b) :

$$A(w) = \frac{\gamma(u)\gamma(v)[A(u)A(v) + B(v)u]}{\gamma(u)\gamma(v)[1 + B(u)v]}$$

En simplifiant par $\gamma(u)\gamma(v)$ et en remplaçant w par sa valeur donnée par l'équation (5a) :

$$A\left[\frac{u + A(u)v}{1 + B(u)v}\right] = \frac{A(u)A(v) + B(v)u}{1 + B(u)v}. \tag{7}$$

Taking $v = 0$ in this equation gives (pour $v = 0$, $A(u) = A(u)A(0) + B(0)u$)

$$A(u)[1 - A(0)] = uB(0). \tag{8}$$

Taking $u = 0$ yields only two solutions, $A(0) = 0$ or $A(0) = 1$. The first case gives $A(u) = uB(0)$. $B(0) \neq 0$ is excluded by reflection invariance (6); then $A(u) = 0$. But (5d) becomes $A(w) = B(w)u$, so that $B(w) = 0$: this is a case of complete degeneration to only one efficient variable since $t' = 0$ for any u, which can thus be excluded, since we are looking for two-variable transformations. We are left with $A(0) = 1$, which implies $B(0) = 0$, and the existence of a neutral element is demonstrated. Let us now take $v = -u$ in (7) after accounting for (6), and introduce a new even function $F(u) = A(u) - 1$, which verifies $F(0) = 0$. We obtain

$$2F(u)\frac{1 + F(u)/2}{1 - uB(u)} = F\left[\frac{uF(u)}{1 - uB(u)}\right] \tag{9}$$

We shall now use the fact that B and F are continuous functions and that $B(0) = 0$. This implies that there exists $\eta_0 > 0$ such that in the interval $-\eta_0 < u < \eta_0$, $1 - uB$ and $1 + F/2$ become bounded to $k_1 < 1 - uB(u) < k_2$ and $k_3 < 1 + F(u)/2 < k_4$ with k_1, k_2, k_3 and $k_4 > 0$.
Donc, pour ε petit, l'équation (9) devient :

$$2F(u) = F[uF(u)]$$

C'est immédiat, il suffisait d'imaginer que l'on s'approche de $u = 0$ dans l'équation (7).
Soit alors $u_0/|F(u_0)| < \varepsilon$ avec $\varepsilon \ll 1$
On a $2F(u_0) = F[u_0(F(u_0)]$
On pose $u_1 = u_0F(u_0)$, donc $u_1 \ll u_0$ puisque $F(u_0) \ll 1$
Donc $F(u_1) = 2F(u_0)$
Comme u_1 est plus petit que u_0, on peut recommencer avec u_1 :
$2F(u_1) = F[u_1(F(u_1)]$
On pose $u_2 = u_1F(u_1) = u_0F(u_0) \times 2F(u_0) = u_0F^2(u_0) \times 2$,
Donc $F(u_2) = 2F(u_1) = 2^2F(u_0)$
$2F(u_2) = F[u_2(F(u_2)]$
On pose $u_3 = u_2F(u_2) = u_0F^2(u_0) \times 2 \times 2^2F(u_0) = u_0F^3(u_0) \times 2^3$,

$$\begin{aligned} F(u_3) &= 2F(u_2) = 2^3F(u_0) \\ \vdots u_p &= u_0F^P(u_0) \times 2^{(1+2+\cdots+p-2+p-1)} \\ &= u_0F^P(u_0)2^{p(p-1)/2} \\ F(u_p) &= 2^pF(u_0) \end{aligned}$$

En prenant la valeur absolue des deux dernières égalité :

$$\begin{aligned} |u_p| &= |u_0||F^p(u_0)|2^{p(p-1)/2} \\ |F(u_p)| &= |F(u_0)|2^p \end{aligned}$$

Pour u_0 donné, on pose que la fonction F prend la valeur $F(u_0) = F_0$.
On peut toujours écrire F_0 sous la forme $F_0 = 2^{-n}$ avec $n > 0$
car $F(u_0) < \varepsilon \ll 1$.
Ceci fixe la valeur de n à :

$$n = -\log_2 F_0$$

Avec cette notation, on a :

$$\begin{aligned} |u_p| &= |u_0||2^{-np}|2^{p(p-1)/2} \\ &= |u_0|2^{-np}2^{p(p-1)/2} \; car \; 2^x > 0 \; \forall x \in \mathbb{R} \\ &= |u_0|2^{p[(p-1)/2-n]} \\ |F(u_p)| &= 2^p2^{-n} \\ &= 2p - n \end{aligned}$$

Lorsque le rang p devient supérieur à n, on a :

$$|F(u_p)| = 2^{p-n} > 1 \gg \varepsilon$$

A noter que lorsque le rang p est égale à Int$[n]$, on a :

$$|F(u_p)| = 2^{Int[n]-n} > \frac{1}{2} \gg \varepsilon$$

Et, lorsque le rang p est inférieur à $2n+1$, on a $\frac{p-1}{2} < n$ et :

$$|u_p| < |u_0|$$

Donc $\forall u_0$ / $|F(u_0| < \varepsilon$, on peut toujours trouver une valeur u_p de u telle que l'on ait $|F(u_p)| > \varepsilon$ et en même temps $|u_p| < |u_0|$. Il suffit en fait que $n < p < 2n+1$. La fonction F ne peut donc pas tendre de façon continue vers 0. F doit par conséquent être constante, et comme $F(0) = 0$ on a $F(u) = 0 \forall u$.
Donc

$$A(u) = 1. \tag{10}$$

As a consequence (7) becomes $B(u)v = B(v)u$, a relation which finally constrains the B function to be

$$B(v) = \kappa v, \tag{11}$$

where κ is a constant.
En effet, $B(u)/u = B(v)/v$ et comme $u \neq v$ on a $B(v) = C^{st}$.
At this stage of our demonstration, the law of transformation of velocities is already fixed to the Einstein-Lorentz form (équation (5a)):

$$w = \frac{u+v}{1+\kappa uv}, \tag{12}$$

and it is easy to verify that a full group law is obtained, i.e. the existence of an identity transformation and an inverse transformation is demonstrated rather than postulated. Consider now the γ factor. It satisfies the condition (équation (5b)):

$$\gamma\left(\frac{u+v}{1+\kappa uv}\right) = \gamma(u)\gamma(v)(1+\kappa uv). \tag{13}$$

Let us consider the case $u = -v$. Equation (13) reads $\gamma(0) = \gamma(v)\gamma(-v)(1-\kappa v^2)$. For $v = 0$ it becomes $\gamma(0) = [\gamma(0)]^2$ implying $\gamma(0) = 1$, (ou $\gamma(0) = 0$ mais alors $x' = t' = 0$. On note que $\gamma(0) = 1$ et l'élément neutre est démontré) and we get (toujour pour $u = -v$):

$$\gamma(v)\gamma(-v) = \frac{1}{1-\kappa(v^2)}, \tag{14}$$

The final step to the Lorentz transformation is straightforward from reflection invariance, which implies that $\gamma(v) = \gamma(-v)$ (see Eq. 6) and fixes the γ factor in its Lorentz-Einstein form:

$$\gamma(v) = \frac{1}{\sqrt{1 - \kappa v^2}}. \tag{15}$$

The case $\kappa < 0$ yields a non-ordered group (applying two successive positive velocities may yield a negative one), and we are left with only two physical solutions, the Galileo ($\kappa = 0$) and Lorentz ($\kappa = c^{-2} > 0$) groups. Three of their properties (existence of a neutral element and of an inverse element, commutativity in case of one space dimension) have not been postulated, but deduced from our initial axioms.

Let us end this section with a brief but important comment. We have shown that, once linearity is assumed, the Lorentz transformation may be obtained through only the postulates of internal composition law and reflection invariance. Linearity is not a constraint by itself: indeed it corresponds to the simplest-possible choice (i.e. when searching for a transformation which would satisfy a given law, one may first look for a linear one, and then for non-linearity only in case of failure, or later as a generalization). With regard to the other two postulates, they may be seen as a *direct translation of the Galilean principle of relativity*. Indeed the hypothesis that the composed coordinate transformation ($K \to K''$) and the transformation in the reversed frame ($-K \to -K'$) must keep the same form as the initial one ($K \to K'$) is nothing but an application of the Galilean principe of relativity ("the laws of nature must keep the same form in different inertial reference systems") *to the laws of coordinate transformation themselves*, which are clearly part of the laws to which the principle should apply. So the general solution to the problem of inertial motion, without adding any postulate to the way it might have been stated in the Galileo and Descartes epoch, is actually Einstein's special relativity, of which Galilean relativity is a special case ($c = \infty$).

[1] Levy-Leblond, J.M., 1976, *Am. J. Phys.* **44**, 271
[2] Bacry, H., & Levy-Leblond, 1968, *J. Math. Phys.* **9**, 1605