

Additivity, rapidity, relativity

Jean-Marc Lévy-Leblond¹

Physique Théorique, Université Paris VII, France

Jean-Pierre Provost²

Physique Théorique, Université de Nice, France

(Received 22 May 1978; accepted 8 August 1979)

A simple and deep standard mathematical theorem asserts the existence, for any one-parameter differentiable group, of an additive parameter, such as the angle for rotations and the rapidity parameter for Lorentz transformations. The importance of this theorem for the applications of group theory in physics is stressed, and an elementary proof is given. The theorem then is applied to the construction from first principles of possible relativity groups.

Introduction

Special relativity usually is introduced through the Lorentz transformation formulas

$$\begin{aligned}x' &= \frac{x - vt}{\sqrt{1 - v^2}} \\t' &= \frac{t - vx}{\sqrt{1 - v^2}},\end{aligned}\tag{1}$$

expressed in terms of the relative velocity v of the two reference frames (our units are such that $c = 1$). These formulas, it must be admitted, are not very elegant and often seem complicated to the student. They imply new and strange properties for the velocity: existence of a limit velocity c , and the curious composition law:

$$v_{12} = (v_1 + v_2)/(1 + v_1 v_2).\tag{2}$$

At a later stage, the student, in modern courses at least, is introduced to the *rapidity* parameter φ defined by

$$v = \tanh \varphi\tag{3}$$

in terms of which the Lorentz formulas take the simple form

$$\begin{cases}x' = \cosh \varphi x - \sinh \varphi t \\t' = \cosh \varphi t - \sinh \varphi x,\end{cases}\tag{4}$$

and the composition law becomes

$$\varphi_{12} = \varphi_1 + \varphi_2.\tag{5}$$

The additivity of rapidity exhibits at once the fundamental group property of Lorentz transformations.

Rapidity is not merely a convenient parameter. Its introduction in the teaching of Einsteinian relativity stresses the need to replace the Galilean velocity by two separate concepts: "velocity" v , as expressing the time rate of change of position, and

"rapidity" φ , as the natural (additive) group parameter.³ Emphasizing the distinction helps in eliminating difficulties. For instance, students often ask whether the existence of a limit velocity c (for material particles) does not imply the existence of a "limit reference frame", which would contradict the relativity principle. They also wonder how the consideration of velocities greater than c (for "unmaterial" objects, such as shadows—or for tachyons) can be consistent with Einsteinian relativity. These pseudoparadoxes disappear when it is realized that the intrinsic parameter of the group law, expressing the transformation from one reference frame to another, is rapidity, which has no limit and may increase indefinitely, while, conversely, there is no rapidity, i.e., no change of reference frame, associated with superluminal velocities.⁴ Beyond its pedagogical utility, rapidity in the recent years has become a common and efficient tool in particle physics.

It seems worthwhile, for these reasons, to introduce rapidity as soon as possible in the teaching of relativity, namely, at the start. There is no need to go through the expressions of Lorentz transformations using velocity, and then to "discover" the elegant properties of rapidity as if they resulted from some happy and unpredictable circumstance. Indeed, there exists a deep, if simple, mathematical theorem establishing in advance the necessity of such a parameter. The theorem asserts the existence of an additive parameter for any one-parameter group (differentiable and connected). Unfortunately, this theorem, despite its importance and simplicity, is usually introduced in connection with the general study of Lie groups and their algebras, requiring rather elaborate mathematics, unfit to the needs of elementary physics. It is the purpose of the present paper to give a simple proof of the theorem, to illustrate it by elementary examples, and to use it as a basis for the derivation of relativity theory. Besides this specific use, the present considerations may also be considered as an introduction to some important ideas of the group theory which play an essential role in contemporary theoretical physics.

The paper is organized as follows. In Sec. I, the theorem is stated and proved. Section II is devoted to some examples. Section III uses the theorem to present a derivation of relativity theory, for one space dimension, from first principles, alternative to a recent proposal. The new derivation allows for a straightforward generalization to three-dimensional space, which is presented in Sec. IV.

I. ADDITIVE PARAMETER THEOREM

We consider a "one-parameter group", i.e., a group G , the elements of which are labeled by real numbers. In more rigorous terms, the group G is supposed to be in one-to-one correspondence with some connected subset S of R , which means that the set of parameters S is just one single piece of the real line, finite or not. For instance, $S =] - \infty, +\infty[$, if G is the translation group with the length of the translation as a parameter, and $S =] - c, +c[$ if G is the Lorentz group with the velocity as a parameter. We denote by $g(x)$, $x \in S$, the generic element of G . The group structure requires the existence of a composition law: the product $g(x)g(y)$ of two elements of G is an element of G labeled by a parameter which is a function of x and y and will be denoted by $x * y$.

$$g(x)g(y) = g(x * y) \tag{6}$$

We may as well identify each element of G with its parameter x , considering the group as the set S endowed with the composition law $*$. This law of course satisfies the

group axioms, that is

1) associativity:

$$(x * y) * z = x * (y * z), \quad (7)$$

2) existence of a neutral element (identity) e :

$$e * x = x * e = x, \quad (8)$$

3) existence of an inverse \bar{x} for each element x :

$$x * \bar{x} = \bar{x} * x = e. \quad (9)$$

We now suppose that the composition law has some smoothness property. More precisely, we ask that the function f_x of left multiplication by x ,

$$f_x(y) \stackrel{df}{=} x * y, \quad (10)$$

possesses a first-order derivative, and we require it to differ from zero at the neutral element

$$f'_x(e) \neq 0. \quad (11)$$

Writing the associativity property as

$$f_{x*y}(z) = f_x[f_y(z)] \quad (12)$$

and differentiating it with respect to z at the point $z = e$, we get, taking into account $f_y(e) = y$:

$$f'_{x*y}(e) = f'_x(y)f'_y(e). \quad (13)$$

The above requirement (1.6) therefore implies that $f'_x(y) \neq 0$ for all $y \in S$, so that we have $f_x(y_1) \neq f_x(y_2)$ and $x * y_1 \neq x * y_2$ whenever $y_1 \neq y_2$. For this reason we call a parametrization with the above smoothness properties (10)-(11) an "essential" parametrization.

Of course the labeling of the group is arbitrary. If we consider the Lorentz group, for instance, we can equally well characterize a given Lorentz transformation by its velocity v , or its rapidity φ or its "momentum" $v(1-v^2)^{-1/2} = \sinh \varphi$. Whether there is some *natural* parametrization for the abstract group G , precisely is the question we consider here. A general change of parametrization may be defined by some differentiable function λ such that λ^{-1} exists as well. The group elements are now labeled by $\xi = \lambda(x)$, $\eta = \lambda(y)$, etc., and the composition law is given by

$$\xi o \eta = \lambda[\lambda^{-1}(\xi) * \lambda^{-1}(\eta)]. \quad (14)$$

The additive parameter theorem, which we now prove, mainly asserts that it is always possible to choose a suitable λ such that the o law simply is the ordinary addition of real numbers; that is, a function λ such that

$$\lambda(x * y) = \lambda(x) + \lambda(y). \quad (15)$$

Differentiating (15) with respect to y , we obtain the following equation for λ :

$$\lambda'(x * y) \cdot f'_x(y) = \lambda'(y), \quad (16)$$

and remark that the identity (13) provides us at once with a solution for this equation, namely:

$$\lambda'(x) = f'_x(e)^{-1}. \quad (17)$$

The function λ defined by

$$\lambda(x) = \int_e^x [f'_t(e)]^{-1} dt, \quad (18)$$

therefore obeys the additive property (15). The lower bound e in the integral is dictated, of course, by the requirement that $\lambda(e) = 0$, an immediate consequence of the addition law.

Is this parametrization unique? Let us denote by ξ, η , etc. the additive parameters. Suppose that there exists another additive parametrization ξ', η' , etc., given in terms of the first one by some function μ : $\xi' = \mu(\xi)$, it must obey

$$\mu(\xi + \eta) = \mu(\xi) + \mu(\eta). \quad (19)$$

It results that μ is a linear function

$$\mu(\xi) = k\xi, \quad (20)$$

i.e., the additive parametrization is unique up to a multiplicative constant. Indeed, such a constant could be included in the expression (18) without altering its property (15).

In conclusion, what we have proven is that under the hypothesis of essential parametrization, any one-dimensional connected differentiable group is isomorphic to the additive group of real numbers. In particular, this result implies that the group is abelian, which is far from evident *a priori*.⁵

Now there are important groups without a differentiable essential parametrization. The rotation group in the plane is such a case; we may choose an additive parameter, the angle, but either we restrict its range to $[0, 2\pi]$ and lose differentiability, or take the entire set of real numbers but lose uniqueness of the parametrization. However, when the parametrization is not essential but if $f'_x(e)$ differs from zero in a neighborhood of the identity, the isomorphism exists in such a neighborhood. The general result, which we shall not prove, is that any connected one-dimensional differentiable group is isomorphic either to \mathbf{R} (the real line) or to $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ (the circle, as the plane rotation group). One may even abandon the hypothesis of differentiability and only ask for local measurability of the function f_x . A nice counterexample exhibiting the limits of the theorem is the following. Consider a two-parameter group, not necessarily abelian, for instance the one-dimensional translation-dilatation group. Since, setwise, \mathbf{R}^2 has the same cardinality as \mathbf{R} , we may always replace the two real parameters of the group by a single one using some bijection (such as the old trick of combining two real numbers in decimal form by writing a single number the odd and even digits of which are those of the two numbers). The group now is a "one-parameter" group (without any physical significance, of course). However, since no one-to-one mapping $\mathbf{R}^2 \leftrightarrow \mathbf{R}$ can be a measurable one, the conditions of the theorem are not fulfilled and there is no overall additive parameter—and fortunately so, since the group is not abelian!

II. SOME EXAMPLES

A. Consider the multiplicative group \mathbf{R}^\times of (positive) real numbers. We have here

$$f_x(y) = xy, \quad e = 1, \quad f'_t(e) = t. \quad (21)$$

The additive parameter, the existence of which is guaranteed by the theorem, is explicitly given by formula (17):

$$\lambda(x) = \int_t^x t^{-1} dt = \log x \quad (22)$$

Of course, this is but the definition of the Napierian logarithm function, the main property of which precisely is to realize the isomorphism of the multiplicative group of positive real numbers with the additive group of real numbers.

B. The 2×2 real symmetrical and unimodular matrices form a group; the generic element may be written:

$$M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad a^2 - b^2 = 1, \quad a \leq 1 \quad (23)$$

Considering b as the parameter of the group with $a = (1 + b^2)^{1/2}$, the group law may be written

$$b * b' = (1 + b^2)^{1/2} b' + b(1 + b'^2)^{1/2}. \quad (24)$$

One has

$$\begin{aligned} f_x(y) &= (1 + x^2)^{1/2} y + x(1 + y^2)^{1/2}, \\ e &= 0, \quad f'_t(e) = (1 + t^2)^{1/2}. \end{aligned} \quad (25)$$

An additive parameter exists and is given by

$$\varphi(b) = \int_0^b (1 + t^2)^{-1/2} dt = \arg \sinh b. \quad (26)$$

We have then

$$b = \sinh \varphi, \quad a = \cosh \varphi, \quad (27)$$

so that the group elements are written

$$M(\varphi) = \begin{pmatrix} \cosh \varphi & \sinh \varphi \\ \sinh \varphi & \cosh \varphi \end{pmatrix}. \quad (28)$$

This abstract group can be considered as the Lorentz group for one space dimension, φ being now the rapidity; we will investigate this group from a more physical point of view in the next section.

There is another way to derive the expression (25), avoiding for students the integration (27). It suffices to assume the existence of the additive parameter φ and to express the group law in terms of it:

$$M(\varphi)M(\varphi') = M(\varphi + \varphi'). \quad (29)$$

This matrix equation leads to:

$$\begin{cases} a(\varphi + \varphi') = a(\varphi)a(\varphi') + b(\varphi)b(\varphi') \\ b(\varphi + \varphi') = b(\varphi)a(\varphi') + a(\varphi)b(\varphi') \end{cases} \quad (30)$$

Let us define

$$\begin{cases} u(\varphi) = a(\varphi) + b(\varphi) \\ v(\varphi) = a(\varphi) - b(\varphi) \end{cases} \quad (31)$$

These functions then obey:

$$\begin{cases} u(\varphi + \varphi') = u(\varphi)u(\varphi') \\ v(\varphi + \varphi') = v(\varphi)v(\varphi') \end{cases} \quad (32)$$

so that they are exponential functions. The unimodularity condition $a^2 - b^2 = 1$ writes

$$u(\varphi)v(\varphi) = 1, \quad (33)$$

so that u and v are inverse exponentials:

$$\begin{cases} u(\varphi) = \exp(k\varphi) \\ v(\varphi) = \exp(-k\varphi) \end{cases} \quad (34)$$

yielding in turn

$$\begin{cases} a(\varphi) = \cosh(k\varphi) \\ b(\varphi) = \sinh(k\varphi) \end{cases} \quad (35)$$

in agreement, of course, with (28) (up to the allowed constant factor k).

C. The 2×2 real antisymmetrical unimodular matrices

$$R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a^2 + b^2 = 1 \quad (36)$$

form a group as well, the group of plane rotations. Now, because of the sign ambiguity, $a = \pm(1 - b^2)^{1/2}$, the parametrization by b is not an essential one. However, in a neighborhood of the identity (characterized by $b = 0, a = 1$), the group law reads

$$b * b = (1 - b^2)^{1/2}b' + b(1 - b'^2)^{1/2}. \quad (37)$$

One has

$$\begin{aligned} f_x(y) &= (1 - x^2)^{1/2}y + x(1 - y^2)^{1/2}, \\ e &= 0, \quad f'_t(e) = (1 - t^2)^{1/2} \end{aligned} \quad (38)$$

(It is clear that for $t = 1, f'_t(e) = 0$ indeed.)

The additive parameter is given by

$$\theta(b) = \int_0^b (1 - t^2)^{-1/2} dt = \arcsin b. \quad (39)$$

We have thus

$$b = \sin \theta, \quad a = \cos \theta, \quad (40)$$

and

$$M(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (41)$$

so that the group is identified with the rotation group in the plane, in accordance with the general theorem.

One could also postulate the existence of θ and derive all known properties of the functions a (cosine) and b (sine) from the group law satisfied by the M matrices.

III. RELATIVITY GROUPS FOR ONE-DIMENSIONAL SPACE

We will now use the additive parameter theorem to establish from first principles the theory of relativity. The discussion here is limited to a one-dimensional space; Sec. IV will generalize it to three-dimensional space. We look for space-time transformations, connecting two inertial reference frames, which satisfy the following requirements: (i) they preserve the homogeneity of space-time; (ii) they form a group; (iii) they are compatible with space reflexion; and (iv) they allow for some notion of causality. It has already been shown elsewhere⁶ that such hypotheses are sufficient to derive the Lorentz transformations (and their degenerate cousins, the Galilei transformations). In particular there is no need for any postulate dealing with the constancy of the speed of light. The reader is referred to previous work for the discussion of the physical significance of the new postulates.⁶

The homogeneity condition (i) is equivalent to the linearity of the transformations in space-time. We thus may write these transformations in matrix form

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} a(\varphi) & b(\varphi) \\ c(\varphi) & d(\varphi) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = M(\varphi) \begin{pmatrix} t \\ x \end{pmatrix} \quad (42)$$

where we suppose the transformation to depend upon a single parameter (see a discussion of this point in the first paper of Ref. 4) which we choose as an additive one, according to the fundamental theorem, so that the matrices $M(\varphi)$ obey

$$\begin{cases} M(\varphi + \varphi') = M(\varphi)M(\varphi') & (a) \\ M^{-1}(\varphi) = M(-\varphi) & (b) \\ M(0) = I. & (c) \end{cases} \quad (43)$$

If we change the direction of the space axis, the transformation labeled by φ becomes represented by the matrix

$$\hat{M}(\varphi) = \begin{pmatrix} a(\varphi) & -b(\varphi) \\ -c(\varphi) & d(\varphi) \end{pmatrix}. \quad (44)$$

The set of these matrices of course form a group isomorphic to the original one. The condition of symmetry under space reflexion now requires the two groups to be identical (and not only isomorphic); namely, there must exist some parameter $\check{\varphi}$, depending on φ , such that

$$\hat{M}(\varphi) = M(\check{\varphi})$$

with

$$\check{\varphi} = \chi(\varphi). \quad (45)$$

Now since φ is an additive parameter for the group of matrices \hat{M} , it is clear that χ must be an additive function of φ . Indeed, from (45) one derives

$$\begin{aligned} M[\chi(\varphi + \varphi')] &= \hat{M}(\varphi + \varphi') = \hat{M}(\varphi) + \hat{M}(\varphi') \\ &= M[\chi(\varphi)] + M[\chi(\varphi')] \\ &= M[\chi(\varphi) + \chi(\varphi')]. \end{aligned} \quad (46)$$

It results that

$$\tilde{\varphi} = \kappa\varphi. \quad (47)$$

Now, changing twice the orientation of the space axis, we must recover the original parametrization, so that $\kappa^2 = 1$. The case $\kappa = 1$ is trivial, since the equation $M(\varphi) = \hat{M}(\varphi)$ leads to $b = c = 0$, that is, no relationship between space and time. We are now left with the condition

$$\hat{M}(\varphi) = M(-\varphi) \quad (= M^{-1}(\varphi)). \quad (48)$$

According to (44), the last relationship yields the parity properties of the functions a , b , c , and d :

$$\begin{aligned} a(-\varphi) &= a(\varphi) & b(-\varphi) &= -b(\varphi) \\ c(-\varphi) &= -c(\varphi) & d(-\varphi) &= d(\varphi) \end{aligned} \quad (49)$$

Taking into account the equalities

$$\det \hat{M}(\varphi) = \det M(\varphi) \quad \text{and} \quad \det M(-\varphi) = [\det M(\varphi)]^{-1},$$

this relationship also yields the unimodularity property of the M matrices

$$\det M(\varphi) = 1. \quad (50)$$

Therefore, (48) becomes

$$\begin{pmatrix} a(\varphi) & -b(\varphi) \\ -c(\varphi) & d(\varphi) \end{pmatrix} = \begin{pmatrix} d(\varphi) & -b(\varphi) \\ -c(\varphi) & a(\varphi) \end{pmatrix}. \quad (51)$$

From (51) we infer

$$a = d, \quad a^2 - bc = 1. \quad (52)$$

Let us now consider the multiplication law (43). In particular we can write

$$a(\varphi + \varphi') = a(\varphi)a(\varphi') + b(\varphi)c(\varphi'). \quad (53)$$

Since this law is commutative, we must have the equality

$$b(\varphi)c(\varphi') = b(\varphi')c(\varphi) \quad (54)$$

or

$$\frac{c(\varphi)}{b(\varphi)} = \frac{c(\varphi')}{b(\varphi')} = C^{st}, \quad (55)$$

so that b and c are two proportional functions—unless one is zero. The proportionality constant can be adjusted to 1 or -1 . Indeed, changing the space scale by a factor h :

$$x \rightarrow hx, \quad (56)$$

corresponds, after (42) to the following transformations of the functions b and c and their quotient

$$b \rightarrow h^{-1}b, \quad c \rightarrow hc, \quad \frac{c}{b} \rightarrow h^2 \frac{c}{b}. \quad (57)$$

Four cases therefore can be distinguished

① $b = c$: The M matrices are unimodular matrices of the type

$$M(\varphi) = \begin{pmatrix} a(\varphi) & b(\varphi) \\ b(\varphi) & a(\varphi) \end{pmatrix}. \quad (58)$$

We recover the *Lorentz group* (see Sec. II B)

② $b = -c$: The M matrices are unimodular matrices of the type

$$M(\varphi) = \begin{pmatrix} a(\varphi) & b(\varphi) \\ -b(\varphi) & a(\varphi) \end{pmatrix}. \quad (59)$$

The group (see Sec. II C) is a *rotation group*—in space time!

③ $b = 0$: The modularity property (52) implies that $a = 1$. The M matrices are of the type

$$M(\varphi) = \begin{pmatrix} 1 & 0 \\ c(\varphi) & 1 \end{pmatrix}. \quad (60)$$

The group law yields

$$c(\varphi + \varphi') = c(\varphi) + c(\varphi') \quad (61)$$

or

$$c(\varphi) = \gamma\varphi. \quad (62)$$

The corresponding group is the *Galilei group* with a dimensionless parameter φ (and γ an arbitrary velocity unit).

④ $c = 0$: Proceeding as in ③ we obtain

$$M(\varphi) = \begin{pmatrix} 1 & \beta\varphi \\ 0 & 1 \end{pmatrix}, \quad (63)$$

which is the definition of the *Carroll group*.

We now introduce a requirement of causality⁶ according to which there exist certain pairs of events such that their time interval Δt keeps the same sign in all reference frames, i.e., remains invariant under transformations with any φ . This condition suffices to dismiss cases ② and ④. The Lorentz and Galilei groups then remain as the only possible physical groups of relativity in space time.

IV. RELATIVITY GROUPS FOR THREE-DIMENSIONAL SPACE

The additive parameter theorem allows for an easy generalization of the preceding derivation to three-dimensional space. We write the space transformation as

$$\begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} = \begin{pmatrix} a(\varphi) & b(\varphi) \\ c(\varphi) & d(\varphi) \end{pmatrix} \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = M(\varphi) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \quad (64)$$

where a , b , c , d now are, respectively, 1×1 , 1×3 , 3×1 and 3×3 matrices. All expressions of Sec. III up to (50) then remain valid. However, the previous algebraic derivation, which needs no differentiation technique, becomes cumbersome for three-dimensional space and we prefer an infinitesimal approach, which indeed is closer to advanced methods relying on Lie algebras (this method, of course, also works in the preceding one-dimensional case).

Consider the multiplication law (43). Deriving with respect to φ' and putting $\varphi' = 0$, we obtain the following equality

$$M'(\varphi) = M(\varphi)M'(0), \quad (65)$$

where the derived matrix is

$$M'(\varphi) = \begin{pmatrix} a'(\varphi) & b'(\varphi) \\ b'(\varphi) & d'(\varphi) \end{pmatrix}. \quad (66)$$

For $\varphi = 0$, according to (49), we see that

$$M'(0) = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \quad (67)$$

where

$$\beta = b'(0), \quad \gamma = c'(0) \quad (68)$$

(do not forget that β and γ are 1×3 and 3×1 matrices, that is, a "line" and a "column" vector, respectively). Relationship (65) now yields a set of differential equations

$$\begin{aligned} a' &= b\gamma, & b' &= a\beta \\ c' &= d\gamma, & d' &= c\beta \end{aligned} \quad (69)$$

from which we deduce the equation

$$a'' = \beta\gamma a. \quad (70)$$

($\beta\gamma$ is the scalar product of the vectors β and γ .) The function a therefore is of exponential type. The specific solution to be selected is dictated by the parity property (49) and the initial condition (43c). Similar argument apply to the initial computation of b , c , d . The nature of the solutions depends on the sign of $\beta\gamma$, or its vanishing. Since a scale change in the additive parameter multiplies $\beta\gamma$ by a positive number, we are led to consider the following cases. ① $\beta\gamma = 1$. We obtain

$$M(\varphi) = \begin{pmatrix} \cosh \varphi & (\sinh \varphi)\beta \\ (\sinh \varphi)\gamma & 1 + (\cosh \varphi - 1)\gamma\beta \end{pmatrix}. \quad (71)$$

② $\beta\gamma = -1$. It suffices to replace in (71) the hyperbolic functions by ordinary sine and cosine functions.

③ $\beta\gamma = 0$. Integration yields:

$$M(\varphi) = \begin{pmatrix} 1 & \varphi\beta \\ \varphi\gamma & 1 + \frac{1}{2}\varphi^2\gamma\beta \end{pmatrix}. \quad (72)$$

The causality condition now eliminates case ② and imposes $\beta = 0$ in case ③. We are left with two cases only:

③ $\beta = 0$, leading to the three-dimensional Galilei transformations

$$M(\varphi) = \begin{pmatrix} 1 & 0 \\ \varphi\gamma & 1 \end{pmatrix}, \quad (73)$$

where the vector γ defines the direction of the Galilean boost.

③ case ①. To put the matrix (71) in a more familiar form, let us perform an arbitrary change of space coordinates through some matrix T . The transformation matrix M now becomes:

$$\begin{aligned} & \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} M(\varphi) \begin{pmatrix} 1 & 0 \\ 0 & T^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \cosh \varphi & (\sinh \varphi)\beta T^{-1} \\ (\sinh \varphi)T\gamma & [1 + (\cosh \varphi - 1)]T\gamma\beta T^{-1} \end{pmatrix}. \end{aligned} \quad (74)$$

It is easily seen that a matrix T always exists such that βT^{-1} is the transposed of the matrix $T\gamma$. With convenient units we now obtain

$$M(\varphi) = \begin{pmatrix} \cosh \varphi & (\sinh \varphi)n^t \\ (\sinh \varphi)n & [1 + (\cosh \varphi - 1)]nm^t \end{pmatrix}, \quad (75)$$

which is a standard expression for the Lorentz transformation with rapidity φ , in the direction of the unit vector n .

The question can be asked why we only found genuine space-time transformations (Lorentz or Galilei) as possible transformations between reference frames. Indeed, it is clear that purely spatial transformations, namely rotations in three-dimensional space, exist as well, and should show up in our derivation of relativity groups. The answer is to be found in the dismissal, after equation (47), of the solution $\kappa = 1$.

ACKNOWLEDGMENTS

One of the author (J.-M. L.-L.) wishes to thank the Département de Physique, Université de Montréal, where part of the work was done, for its hospitality. Prof. J.-P. Serres graciously furnished us with the counter-example at the end of Sec. I.

¹Present address: Laboratoire de Physique Théorique et Hautes Energies, Université Paris VII (Tour 33), 2 Place Jussieu, 75221 Paris Cedex 05, France.

²Present address: Laboratoire de Physique Théorique, Université de Nice, Parc Valrose, 06034 Nice Cedex, France.

³See, for instance, J.-M. Lévy-Leblond, Riv. Nuovo Cimento **7**, 187 (1977) and J.-M. Lévy-Leblond (to be published).

⁴When $v > 1$, formula (3) leads to complex and indeed undetermined values of φ . Another way to understand the restriction $v < 1$ for velocities of reference frames, without invoking the reality of φ , is to remark that the addition law (2) is a group law only if the v 's are less than one. The importance of the group law hypothesis will be developed in Sec. III.

⁵Let us also note that the existence of an inverse has not been invoked in the demonstration. Therefore, a one-dimensional differentiable connected semi-group is isomorphic to the additive semi-group of positive real numbers.

⁶J.-M. Lévy-Leblond, Am. J. Phys. **44**, 271 (1976), A. R. Lee and T. M. Kalotas, Am. J. Phys. **43**, 434 (1975), and additional references in these papers.