Lévy-Leblond ab omni naevo vindicatus

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Dear Colleagues,

The article "One more derivation of the Lorentz transformation" by Jean-Marc Lévy-Leblond [1] is, in my opinion, an excellent starting point for an introductory course in special relativity. His derivation of the Lorentz (and Galilean) transformation is well structured and shows clearly the essential points of the theory (or *theories*) of relativity. He obtains the transformations only from the principles of relativity and of inertia, the homogeneity and isotropy of space-time, the group structure and the causality requirement. (There exist derivations with even less basic assumptions (see [2]), but [1] should be preferred from a didactical point of view.) The hypothesis of the invariance of the speed of light is *not* used!

The seminal works on this alternative approach appeared in German reviews since 1910 (see [3] and [4] for references and historical background). But, astonishingly, Albert Einstein seems to have never commented or endorsed the new derivation, although it could have been important to disentangle the theory of special relativity from electromagnetism (or any other theory) in view, for example, of the construction of the theory of general relativity or just for the sake of conceptual simplification. Maybe this little mystery in history of science is worth an investigation, as suggested by Lévy-Leblond [5].

[1] is restricted to the one-dimensional case, but an extension to the most general case is easy and leads to the representation of the full Lorentz group without the use of too complex or abstract mathematics. The procedure I followed is outlined in [6], where the calculations (including the ones from [1]) are performed with the fabulous computer algebra system *Mathematica* from Wolfram Research.

As far as I know, the only major weak point in the mathematical treatise in [1] is the derivation of the formula (30) $\lambda(v) = 1$, which "is not a completely rigorous proof" as the author admits in a note. In the following I try to derive this result in a more rigorous way using only basic calculus techniques (as can be found in any introductory calculus text like the rigorous and impeccably well-written [7]). First I try to proof a general analysis theorem (which is perhaps an interesting stand-alone calculus exercise) and then, in the corollary, I apply the result to the case in [1].

If your students are not ready yet to follow the formal proof or you want to focus on physics rather then on mathematics (which is never a bad idea in a *physics* course), then you may try to convince them with the easy informal proof given at the end of this article. (Actually it was the informal proof which inspired the formal one.)

Theorem. *Let $f: \mathbb{R} \to \mathbb{R}$ be a real-valued function with following proprieties:*

a) f is continuous on $\mathbb{R}$,

b) f possesses an inverse $f^{-1}: \mathbb{R} \to \mathbb{R}$,

c) $f(x) = f^{-1}(x)$ everywhere on $\mathbb{R}$,

d) $f(-x) = -f^{-1}(x)$ everywhere on $\mathbb{R}$.

Then $f(x) = x$ and $f(x) = -x$ are the unique functions which satisfy the proprieties a) - d).

Observations. The graph of a function is in general symmetric with respect to the bisector y = x to the graph of its inverse function when both curves are plotted on the same Cartesian graph. Hence propriety c) implies that the graph of f is symmetric with respect to the bisector y = x. Propriety d) implies that it is also symmetric with respect to the bisector y = - x. (See fig. 1.)

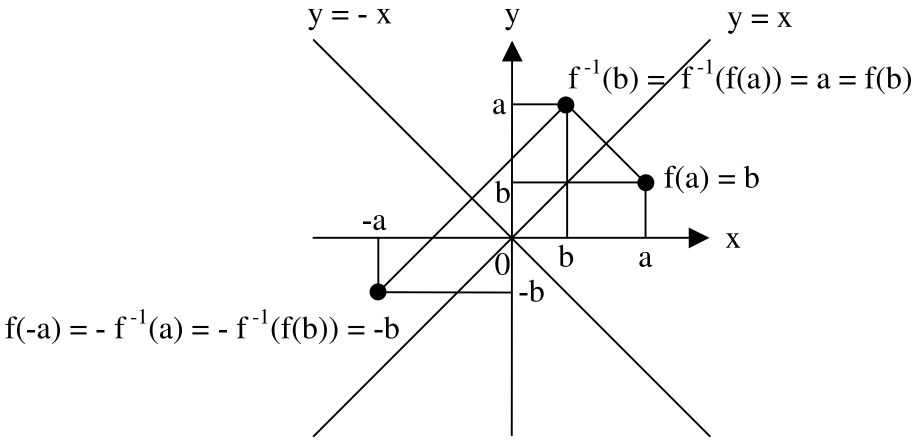

Fig. 1 Symmetries

Proof. It's easily verified that f(x) = x and f(x) = - x satisfy the proprieties a) - d). We have to prove the uniqueness of these solutions.

Because $f(0) = f^{-1}(0) = -f(0)$, we must have f(0) = 0.

We choose some arbitrary x' > 0 and assume initially that f(x') lays in the first quadrant ($x > 0$, $y \geq 0$).

First we assume that f(x') > x' and define the continuous function g(x) = f(x) - x. Let A = {x | g(x) = 0 and x < x'} be the set of all zeros of g(x) minor of x'. A is non-empty ($0 \in A$) and A has an upper bound x', hence for the *completeness axiom* there exist a least upper bound $x_s = \sup(A) \leq x'$. For the *sign permanence theorem on continuous functions* we must have $g(x_s) = 0$, hence $x_s < x'$, $x_s \in A$ and $f(x_s) = x_s$. We have g(x') > 0 and no zeros of g(x) on $(x_s, x']$. If g(x) would change sign on $(x_s, x']$, *Bolzano's theorem* would imply the existence of a zero of g(x) on $(x_s, x']$. Hence g(x) doesn't change sign on $(x_s, x']$, so we must have g(x) > 0 everywhere on $(x_s, x']$, that means f(x) > x on $(x_s, x']$. By reflection of the point (x', f(x')) with respect to the bisector y = x we get the point $(x'', f^{-1}(x''))$ with x'' > x'. The reflection symmetry implies that if $f(x_s) = x_s$ and f(x) > x on $(x_s, x']$, then $f^{-1}(x) < x$ on $(x_s, x'']$. In particular we get $f^{-1}(x') = f(x') < x'$, in contradiction with our prior assumption f(x') > x'. (See fig. 2.)

Now we assume that $0 < f(x') < x'$. By reflection of the point $(x', f(x'))$ with respect to the bisector $y = x$ we get the point $(x'', f^{-1}(x''))$ with $0 < x'' < x'$ and $f^{-1}(x'') = f(x'') > x''$. As shown in the previous paragraph, this leads to a contradiction. (See fig. 3.)

Let's finally assume $f(x') = 0$. We get by reflection of the point $(x', f(x'))$ with respect to the bisector $y = x$ the point $(0, f^{-1}(0))$, hence $x' = f^{-1}(0) = f(0) > 0$ in contradiction with our prior result $f(0) = 0$. (See fig. 4.)

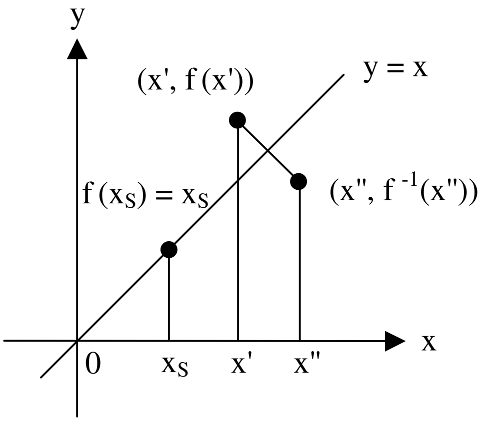

Fig. 2 $f(x') > x'$

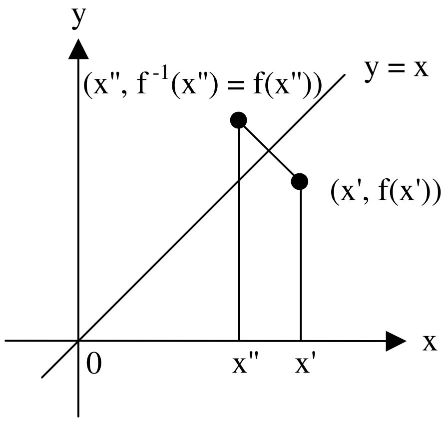

Fig. 3 $0 < f(x') < x'$

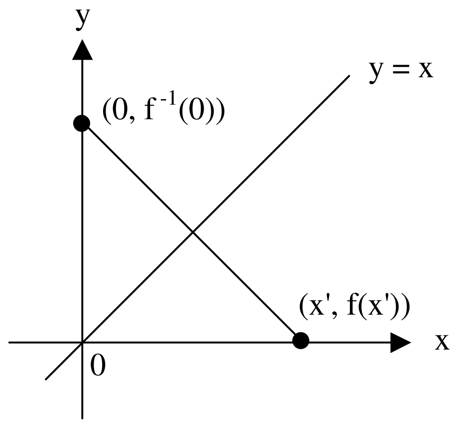

Fig. 4 $f(x') = 0$

We can now conclude that, on the first quadrant, all points of the graph of f must lay on the bisector $y = x$.

We can transform the conditions for f in the second, third and fourth quadrant into the already discussed case of the first quadrant by appropriate reflections with respect to the axes of the Cartesian graph. Then we see that the points of the graph of f must lay on the bisectors $y = x$ or $y = -x$. Since the continuous solutions $f(x) = |x|$ and $f(x) = -|x|$ are ruled out by the proprieties c) and d), we deduce that $f(x) = x$ and $f(x) = -x$ are the unique solutions satisfying all requested proprieties. q.e.d.

Corollary. (I will switch, by now, to the notation and equation numbering in [1].) From definition (23) we know that the continuity of $\zeta(v)$ derives from the continuity of (non-zero) $\lambda(v)$, which, in turn, derives from the requirement of continuity of the transformation formulas (10) as functions of the velocity v. (This is finally the mathematical expression of an experimental fact and may be considered as an independent hypothesis on the transformation formulas.) Proprieties b), c) and d) are implied by (25) and (29). We conclude that $\zeta(v)$ satisfies all proprieties required by the previous theorem and get the two solutions $\zeta(v) = v$ and $\zeta(v) = -v$. From (23) we get $\lambda(v) = v / \zeta(v)$ which is not defined for $v = 0$. But we know that $\lim_{v\to 0} \lambda(v) = \lim_{v\to 0} v / \zeta(v) = 1$, which is satisfied only by $\zeta(v) = v$, hence we get the unique solution $\lambda(v) = 1$. q.e.d.

"Fast & Furious" Proof. The graph of f has to be symmetric with respect to both bisectors of the Cartesian graph (see **Observations**). We have $f(0) = f^{-1}(0) = -f(0)$, hence $f(0) = 0$: the origin is on the graph of f. Now imagine to draw the graph of f starting from the origin as a single curve with no "holes" or "jumps" (continuity!), while three "magic hands" (invoked by Harry P*l*otter…) draw simultaneously the symmetric lines with respect to the bisectors (see fig. 5). This points are part of the graph of f too. If you don't stay straight on the bisectors, you can't avoid to draw the graph of a double-valued relation, but the continuity requirement implies that f is single-valued. Hence all points of the graph of f must lay on the bisectors. The solutions $f(x) = |x|$ and $f(x) = -|x|$ are ruled out by the proprieties c) and d), hence $f(x) = x$ and $f(x) = -x$ are the unique solutions satisfying all requested proprieties. "That's all Folks!"

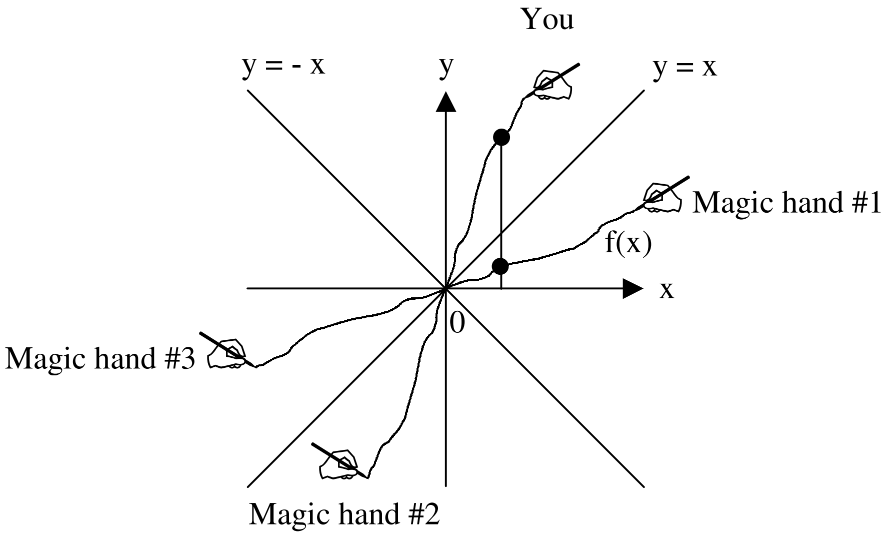

Fig. 5 By magic…

Note: If you drop the requirements a) - d) in 0, you get plenty of alternative solutions (e.g. f(x) = 1/x).

I hope that the present contribution may help to render [1] as self-consistent and rigorous as possible. I would appreciate any criticism about the present work.

[1] Jean-Marc Lévy-Leblond, "One more derivation of the Lorentz transformation", *American Journal of Physics* **44**, 271-277 (1976).
[2] Laurent Nottale, A New Derivation of Lorentz Transformation, *Fractal Space-time and Microphysics*.
[3] Waldemar Sergius von Ignatowsky, "Einige allgemeine Bemerkungen zum Relativitätsprinzip", *Verh. Deutsch. Phys. Ges.* **12**, 788–796 (1910); "Einige allgemeine Bemerkungen zum Relativitätsprinzip", *Phys. Zeitsch.* **11**, 972–976 (1910); "Das Relativitätsprinzip", *Arch. der Math. und Phys. III* **17**, 1–24 (1910); **18**, 17–41 (1911); "Eine Bemerkung zu meiner Arbeit 'Einige allgemeine Bemerkungen zum Relativitätsprinzip'", *Phys. Zeitsch.* **12**, 776-779 (1911).
[4] Jean-Marc Lévy-Leblond, "What if Einstein had not been there? A Gedankenexperiment in Science History", *Proceedings of the 24th International Colloquium on Group Theoretical Methods in Physics*, J. -P. Gazeau et al. (eds), Institute of Physics, London, 2003, note 10.
[1], [2] and [4] can be found at http://o.castera.free.fr/ together with other very interesting material on special relativity.
[5] private communication.
[6] Special_Relativity.ZIP, downloadable at http://mio.discoremoto.alice.it/luigimasciovecchio/. This ZIP-file contains the original editable *Mathematica* notebook and a PDF print-out of the same file in case you haven't the required software. (You can get the free *Mathematica Player* at www.wolfram.com.)
[7] Tom M. Apostol, *Calculus*, Vol. 1: One-Variable Calculus with an Introduction to Linear Algebra, Second Edition (June 1967), Wiley, ISBN-10: 0471000051.