SOME GENERAL REMARKS ON THE PRINCIPLE OF RELATIVITY

W. V. IGNATOWSKY (BERLIN 1910)

When Einstein introduced the principle of relativity, he assumed that the speed of light c was a universal constant, *i.e.* it retained the same value in all frames of reference. Also Minkowski started from the invariant $r^2 - c^2t^2$ in his investigations, although it is to be concluded from his lecture "Space and Time"[1], that he attributed to c the meaning of a universal space-time constant rather than that of the speed of light. I then wondered about the relationships or transformations that emerge if we place the principle of relativity at the center of our study, and whether the Lorentz transformations are the only ones at all that satisfy the principle of relativity.

To answer this question, we will again examine what the principle of relativity teaches us. If we have two reference frames K and K' in translational motion relative to each other, the principle of relativity states that the two frames can be considered equivalent, *i.e.* one can be regarded as at rest while the other is in motion. In other words, we cannot determine absolute motion.

However, if K and K' are equivalent and we can express in frame K any physical quantity E by a function of parameters $a_1, a_2, a_3, \dots$, thus writing

$$E = \varphi(a_1, a_2, a_3, \dots), \tag{1}$$

then the corresponding quantity E' in the frame K' must be expressible by the same function φ of the corresponding parameters $a'_1, a'_2, a'_3, \dots$, *i.e.* it will be

$$E' = \varphi(a'_1, a'_2, a'_3, \dots).$$

If we were to represent E' by the unprimed parameters, e.g.

$$E' = f(a_1, a_2, a_3, \dots) \tag{2}$$

then, since K and K' are equivalent, the equation

$$E = f(a'_1, a'_2, a'_3, \dots) \tag{3}$$

be correct. The equations (1) to (3) form the mathematical formulation of the principle of relativity.

If q denotes the velocity of the frame K' with respect to K, measured from the latter, and q' the velocity of the frame K measured from K', then it must be obvious that

$$q' = -q$$

If we now consider a purely kinematic process, *i.e.* where only x, y, z and t come into consideration, then for instance we can write the following equation

$$x' = \varphi(x, y, z, t, q) \tag{4}$$

and similar equations for y', z', and t'. For x, y, z, and t are to be considered as parameters by which, among others, a physical phenomenon can be described, and from (1) to (3) we see that in general a_1 need not be equal to a'_1.

Although the following calculations are very elementary, to save space, I will only present the reasoning and the final results here, and refer to further details in an article of mine that will soon appear in the Archives of Mathematics and Physics.

We denote by $\vec{u}_0$ the unit vector that indicates the direction of motion of K' with respect to K, then place the X- or X'-axis in this direction, and further assume for simplicity that the X'-axis is the extension of the X-axis.

Since space is assumed to be homogeneous and isotropic, it can be shown from this and for reasons of symmetry that in the equation (4), y and z can only appear implicitly in r, where r is the distance of any point from X-axis. It can further be shown that $r = r'$, and consequently x' cannot depend on r. We can therefore write instead of (4)

$$\left.\begin{aligned} x' &= \varphi(x, t, q) \\ t' &= f(x, t, q) \end{aligned}\right\} \tag{5}$$

and therefore thanks to (2) and (3)

$$\left.\begin{aligned} x &= \varphi(x', t', q') \\ t &= f(x', t', q') \end{aligned}\right\} \tag{6}$$

[1]This journal vol. 10, p. 104, 1909.

If we take the complete differential of (5) and (6), we get

$$\left.\begin{aligned} dx' &= pdx + sdt \\ dt' &= p_1 dx + s_1 dt \end{aligned}\right\} \tag{7}$$

and

$$\left.\begin{aligned} dx &= p'dx' + s'dt' \\ dt &= p_1'dx' + s_1'dt' \end{aligned}\right\} \tag{8}$$

where $p, s, p', s', p_1, s_1, p_1', s_1'$ are eight partial derivatives, which we consider for the moment as unknown functions of x, t, q and x', t', q'. Let D be the determinant

$$D = \begin{vmatrix} p & p_1 \\ s & s_1 \end{vmatrix} = ps_1 - p_1 s$$

It follows from (7) and (8)

$$\left.\begin{aligned} p' &= \frac{s_1}{D}\ ; \qquad & s' &= \frac{-s}{D} \\ p_1' &= \frac{-p_1}{D}\ ; \qquad & s_1' &= \frac{p}{D} \end{aligned}\right\} \tag{9}$$

Now let's take two elements in K and K', dx and dx', so that at relative rest they are of equal lengths. If we now synchronously measure dx' from K (thus $dt = 0$), we obtain

$$dx' = pdx$$

If we synchronously measure dx from K' (thus $dt' = 0$), then

$$dx = p'dx'$$

Now both fames K and K' are equivalent, and dx and dx' have the same length when at relative rest. Therefore, the lengths measured from the two frames must be equal, so

$$p = p' \tag{10}$$

From this and (9) we get

$$p^2 = s_1 s_1'$$

Let us now follow the motion of a material point or a phenomenon in space, and denote its velocity vector by $\vec{v}$ or $\vec{v}'$. Thanks to (7), we can then easily prove that

$$\vec{v}' = \frac{\vec{v} + (p-1)\vec{u}_0\vec{u}_0 \cdot \vec{v} + s\vec{u}_0}{A} \tag{11}$$

where

$$A = s_1 + p_1 \vec{u}_0 \cdot \vec{v}$$

Since $\vec{v}$ is completely arbitrary, it is clear that p, s etc. cannot depend on $\vec{v}$. Let us assume that the moving point is at rest with respect to K'. Then $\vec{v}' = \vec{0}$ and $\vec{v} = q\vec{u}_0$. From this and from (11) we obtain

$$s = -pq$$

By similar considerations to those above

$$s_1 = p$$

so that (7) becomes

$$\left.\begin{aligned} dx' &= pdx - pqdt \\ dt' &= p_1 dx + pdt \end{aligned}\right\}$$

All that remains for us to determine is p_1 and p, since we get the primed quantities from (9). For this purpose, we introduce a third reference frame K'', which moves in the same direction $\vec{u}_0$ with a velocity q_2 measured from K. The velocity of K'' measured from K' is q_1. For couple $K'K''$, we denote the quantities analogous to p_1, p, q by $\bar{p}_1, p, q_1$, and for couple KK'' by p'', p_1'', q_2. We can then easily demonstrate the following relationships :

$$\frac{p_1}{pq} = \frac{\bar{p}_1}{pq_1} = \frac{p_1''}{p''q_2}$$

Since the fractions between them contain independent quantities, we see that these can only be a constant, which we denote by $-n$. So we finally get

$$\left.\begin{aligned} dx' &= pdx - pqdt \\ dt' &= -pqndx + pdt \end{aligned}\right\}$$

From (10) and (9) we also get

$$p^2 = \frac{1}{1 - q^2 n}$$

or

$$p = \frac{1}{\sqrt{1 - q^2 n}} \tag{12}$$

From (12) it follows that n, which can be taken as a universal space-time constant, is the reciprocal square of a velocity, therefore a non-zero positive quantity.

We have obtained transformation equations similar to Lorentz's, except that n is used instead of $\frac{1}{c^2}$. However, the sign of n is not determined because one could just as well place a positive sign in front of q^2n in (12). To determine the numerical value and sign of n, we must now turn to experiment. Since we did not rely on any specific physical phenomenon in the above derivation, it follows that we can determine n based on any phenomenon and should always obtain the same value for n, since n is a universal constant.

For instance, we can synchronously measure the length of a standard meter moving in our frame of reference. If the measurement indicates a contraction, then the sign is negative, and n can be calculated from the contraction factor. However, as we know, the contraction is too small to be measured directly.

Let's now look at the equations of electromagnetism, more specifically at the case of a point charge with uniform motion. In addition to the principle of relativity, we know that the equipotential surface of the electric field of this point charge is a Heaviside ellipsoid for the observer at rest, whose axis ratio is $\sqrt{1-q^2/c^2}$. Based on the principle of relativity, we know that for the observer moving with the point charge the plane surface of the potential is a sphere. In any case, this sphere will appear to the observer in relative motion with respect to the charge, as an ellipsoid having an axial ratio of $\sqrt{1-q^2n}$. Therefore, $\sqrt{1-q^2/c^2}=\sqrt{1-q^2n}$. This gives

$$n=\frac{1}{c^2}$$

This is why the speed of light c is constant in all frames of reference. At the same time, we see that the universal spacetime constant n is determined by the numerical value of c.

The derivation of the above transformation equations clearly shows that optics has lost its special position in the expression of the principle of relativity. In this way, the principle of relativity itself gains in generality, because it no longer depends on a particular physical phenomenon but on a universal constant n.

We can nevertheless grant optics and the equations of electrodynamics a special position, not in relation to the principle of relativity but in relation to other branches of physics, in the sense that it allows us to determine the constant n.

On the other hand, if by transforming the other equations of physics in accordance with the principle of relativity we observe the appearance of the constant n, we should not necessarily conclude that electrical forces are at play, but only that from the point of view of the principle of relativity, space and time leave their mark on all physical phenomena through the constant n.

To illustrate the meaning of n even more clearly, let's consider an analogy from optics, namely the relationship between image and object. From a purely optical-geometric point of view, object and image are interchangeable. Exactly the same is true when we consider a moving measuring rod that appears to us to be contracted. We can say that space and time project this moving measuring rod to us, so that we can only see its image when we assume that the resting measuring rod as being an object.

We can therefore fully subscribe to Minkowski's words, who says in his article "Space and Time"[2] : "Contraction should not be considered as a consequence of resistance in the ether, but simply as a gift from above, as a circumstance concomitant with that of motion", precisely because n is a universal constant.

Finally, I would like to mention the possible speeds from the point of view of the principle of relativity. Let us take the expression (12) for p.

The observed contraction of a line segment moving with K' depends on p. Therefore p cannot be imaginary, so q must always remain less than c. But what does q mean ? q being here the speed of the frame of reference K', it cannot be greater than c. In other words, no frame of reference at rest can have a superluminal relative speed. A reference frame at rest is not a simple mathematical construct; we must imagine a material world with its observers and synchronous clocks. Conversely, we assume that any material point can be at rest. It follows that a material point cannot move at superluminal speed.

(Received September 23, 1910.)

[2]loco citato p. 106.