Special relativity

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```
Print["Revision ", IntegerPart[Date[]]]
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Revision {2010, 5, 2, 18, 2, 22}
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Dear Colleagues,

This is my personal *Mathematica* notebook on special relativity. This document wasn't originally intended for publication, but a few formulas are maybe of interest to you, so here they are. The code seems to work well and I added some comments to make it more understandable. This is *not* an introduction to this field, so use it at your own risk! I have stolen many ideas (with some corrections) from Ladislau Radu "Herleitung der Lorentz-Transformation" (2006, Internet).

VARIA (just a digital garbage can)

A DERIVATION OF THE LORENTZ TRANSFORMATION

In Einstein's theory of special relativity (first published in 1905) the Lorentz transformation converts between two different inertial observers' measurements of space and time, where one observer is in (constant) motion with respect to the other.

☐ *1. Spezielle Lorentztransformation*
(Lorentz transformation for frames in standard configuration)

■ 1.1 Konstruktion der Standardkonfiguration.

(sk = Standardkonfiguration = standard configuration.)

inertial frames O and O' with Cartesian system of coordinates (for space) in both frames
inertial frame O: space-time coordinates (t, x, y, z), velocity of O' measured in O: v
inertial frame O': space-time coordinates (t', x', y', z'), velocity of O measured in O': v'

A) homogeneity of space-time ⇒ space-time transformations between inertial frames are linear. This implies:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} & \Lambda_{03} \\ \Lambda_{10} & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{20} & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{30} & \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{pmatrix} . \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \qquad \text{(Poincaré-Transformation)}$$

```
L = Table[Λ[i, j], {i, 0, 3}, {j, 0, 3}]; Ereignis = (t, x, y, z); A = Table[α[i], {i, 0, 3}];

MatrixForm /@ {L.Ereignis + A}
```

$$\left\{\begin{pmatrix} \alpha[0]+t\,\Lambda[0,0]+x\,\Lambda[0,1]+y\,\Lambda[0,2]+z\,\Lambda[0,3] \\ \alpha[1]+t\,\Lambda[1,0]+x\,\Lambda[1,1]+y\,\Lambda[1,2]+z\,\Lambda[1,3] \\ \alpha[2]+t\,\Lambda[2,0]+x\,\Lambda[2,1]+y\,\Lambda[2,2]+z\,\Lambda[2,3] \\ \alpha[3]+t\,\Lambda[3,0]+x\,\Lambda[3,1]+y\,\Lambda[3,2]+z\,\Lambda[3,3] \end{pmatrix}\right\}$$

B) Assume the coincidence of space-time origins of the two frames. ⇒ $\alpha_i = 0$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} & \Lambda_{03} \\ \Lambda_{10} & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{20} & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{30} & \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{pmatrix} . \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad \text{(Lorentz-Transformation)}$$

```
{(0, 0, 0, 0), "=", L.(0, 0, 0, 0) + A}
```

```
{{{0}, {0}, {0}, {0}}, =, {{α[0]}, {α[1]}, {α[2]}, {α[3]}}}
```

C) Assume that at t = 0 the axis of O overlap the axis of O'. ⇒
(0, x, 0, 0) → (t', x', 0, 0) and (0, 0, y, 0) → (t', 0, y', 0) and
(0, 0, 0, z) → (t', 0, 0, z')
⇒ transformation matrix must be like this:

$$Lsk = \begin{pmatrix} \Lambda[0,0] & \Lambda[0,1] & \Lambda[0,2] & \Lambda[0,3] \\ \Lambda[1,0] & \Lambda[1,1] & 0 & 0 \\ \Lambda[2,0] & 0 & \Lambda[2,2] & 0 \\ \Lambda[3,0] & 0 & 0 & \Lambda[3,3] \end{pmatrix};$$

```
MatrixForm /@ {L.(0, x, 0, 0), L.(0, 0, y, 0), L.(0, 0, 0, z)}

MatrixForm /@ {Lsk.(0, x, 0, 0), Lsk.(0, 0, y, 0), Lsk.(0, 0, 0, z), Lsk.Ereignis}
```

$$\left\{\begin{pmatrix} x\,\Lambda[0,1] \\ x\,\Lambda[1,1] \\ x\,\Lambda[2,1] \\ x\,\Lambda[3,1] \end{pmatrix}, \begin{pmatrix} y\,\Lambda[0,2] \\ y\,\Lambda[1,2] \\ y\,\Lambda[2,2] \\ y\,\Lambda[3,2] \end{pmatrix}, \begin{pmatrix} z\,\Lambda[0,3] \\ z\,\Lambda[1,3] \\ z\,\Lambda[2,3] \\ z\,\Lambda[3,3] \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} x\,\Lambda[0,1] \\ x\,\Lambda[1,1] \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} y\,\Lambda[0,2] \\ 0 \\ y\,\Lambda[2,2] \\ 0 \end{pmatrix}, \begin{pmatrix} z\,\Lambda[0,3] \\ 0 \\ 0 \\ z\,\Lambda[3,3] \end{pmatrix}, \begin{pmatrix} t\,\Lambda[0,0]+x\,\Lambda[0,1]+y\,\Lambda[0,2]+z\,\Lambda[0,3] \\ t\,\Lambda[1,0]+x\,\Lambda[1,1] \\ t\,\Lambda[2,0]+y\,\Lambda[2,2] \\ t\,\Lambda[3,0]+z\,\Lambda[3,3] \end{pmatrix}\right\}$$

■ 1.2 Erstes Relativitätsargument.

We choose the x-axis of O along v.

Two frames obtained by the substitutions y→-z and z→y in O and y'→-z' and z'→y' in O' are connected by the same transformation matrix. This implies:

```
yzRotation = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, -1}, {0, 0, 1, 0}};
Lsk = Lsk /. (Reduce[yzRotation.Lsk == Lsk.yzRotation, Flatten[L]] /. {Equal → Rule, And → List});
MatrixForm /@ {%, Lsk.Ereignis}
```

$$\left\{\begin{pmatrix} \Lambda[0,0] & \Lambda[0,1] & 0 & 0 \\ \Lambda[1,0] & \Lambda[1,1] & 0 & 0 \\ 0 & 0 & \Lambda[2,2] & 0 \\ 0 & 0 & 0 & \Lambda[2,2] \end{pmatrix}, \begin{pmatrix} t\,\Lambda[0,0] + x\,\Lambda[0,1] \\ t\,\Lambda[1,0] + x\,\Lambda[1,1] \\ y\,\Lambda[2,2] \\ z\,\Lambda[2,2] \end{pmatrix}\right\}$$

block structure of the transformation matrix for standard configuration ⇒ 1D-case (t, x) can be treated separately from full 3D-case (t, x, y, z).

The 1D-case is best done *à la Lévy-Leblond.*

1.3 Jean-Marc Lévy-Leblond, "One more derivation of the Lorentz transformation", American Journal of Physics 44, 271-277 (1976).

The principle of relativity is first stated in general terms, leading to the idea of equivalent frames of reference connected through "inertial" transformations obeying a group law. The theory of relativity then is constructed by constraining the transformations through four successive hypotheses: homogeneity of space-time, isotropy of space-time, group structure, causality condition. Only the Lorentz transformations and their degenerate Galilean limit obey these constraints.

Relativity theory, in fact, is but the statement that all laws of physics are invariant under the Poincaré group (inhomogeneous Lorentz group).

Principle of relativity: there exists an infinite continuous class of reference frames in space-time which are physically equivalent. I find it convenient to call "inertial frames" and "inertial transformations" the equivalent reference frames and the transformations connecting them. Let us call "inertial motions" those motions which are obtained from rest by an inertial transformation; an object has an inertial motion in some reference frame if it is at rest in another equivalent frame. (Principle of inertia: a physical object has no absolute state of motion or rest; for instance, a free body (with no "forces" acting on it) is caracterized by an "inertial motion" which is not entirely determined, since it depends on "initial conditions", that is also to say, on the reference frame considered.)

Transformation formulas for the spatiotemporal coordinates (x, t) of an arbitrary event in an inertial frame:

```
<< Utilities`Notation`
Symbolize[x']; Symbolize[t']; Symbolize[xn']; Symbolize[tn'];
```

```
General::spell1 : Possible spelling error: new symbol name "t␣Superscript␣Quote" is similar to existing symbol "x'". More…
General::spell1 :
 Possible spelling error: new symbol name "xn␣Superscript␣Quote" is similar to existing symbol "x'". More…
General::spell :
 Possible spelling error: new symbol name "tn␣Superscript␣Quote" is similar to existing symbols {t', xn'}. More…
```

$\{x' = f[x, t, a_1, a_N], t' = g[x, t, a_1, a_N]\}$

$\{x' = F[x, t, a_1, a_n], t' = G[x, t, a_1, a_n]\}, \; n = N - 2$

$\{x' = F[x, t, a], t' = G[x, t, a]\}$

HYPOTHESIS 1 : HOMOGENEITY OF SPACE - TIME

```
DSolve[{
   ∂x F[x, t, a] == H[a], ∂t F[x, t, a] == -K[a],
   ∂x G[x, t, a] == -M[a], ∂t G[x, t, a] == L[a],
   F[0, 0, a] == 0, G[0, 0, a] == 0},
  {F[x, t, a], G[x, t, a]}, {x, t}];
```

DSolve::overdet : The system has fewer dependent variables than equations, so is overdetermined. More…

```
DSolve[∂x F[x, t, a] == H[a], F[x, t, a], {x, t}];
DSolve[∂t (%[[1, 1, 2]]) == -K[a], C[1][t], t, GeneratedParameters → (Module[{C}, C] &)];
x' = %%[[1, 1, 2]] /. %[[1]];
DSolve[∂x G[x, t, a] == -M[a], G[x, t, a], {x, t}];
DSolve[∂t (%[[1, 1, 2]]) == L[a], C[1][t], t, GeneratedParameters → (Module[{C}, C] &)];
{x', t' = %%[[1, 1, 2]] /. %[[1]]}
Solve[{x' == 0, t' == 0, x == 0, t == 0}, {x'[[1]], t'[[1]]}];
{x', t'} = {x', t'} /. %[[1]]
SolveAlways[{x' == γ (x - ν t), t' == γ (λ t - μ x), v == K[a] / H[a]}, {x, t}]
%[[2]] /. {Rule → Equal, γ → γ[v], λ → λ[v], μ → μ[v], ν → ν[v]} // Flatten;
Solve[%, {H[a], K[a], L[a], M[a]}];
A = {x', t'} = {x', t'} /. %[[1]] // Simplify
```

```
{C$80 + x H[a] - t K[a], C$104 + t L[a] - x M[a]}

{x H[a] - t K[a], t L[a] - x M[a]}

{{L[a] → 0, M[a] → 0, K[a] → 0, H[a] → 0, γ → 0}, {L[a] → γ λ, M[a] → γ μ, K[a] → v γ, ν → v, H[a] → γ}}

{(-t v + x) γ[v], γ[v] (t λ[v] - x μ[v])}
```

HYPOTHESIS 2 : ISOTROPY OF SPACE

```
B = {xn', tn'} = {-x', t'} /. {x → -x, v → u} // Simplify;
A == B // ExpandAll
({Coefficient[A, x], Coefficient[A, t]} // Flatten) ==
  ({Coefficient[B, x], Coefficient[B, t]} // Flatten);
eqn = Table[%[[1, i]] == %[[2, i]], {i, 1, Length[%[[1]]]}];
% // TableForm
Solve[eqn, u]
eqn = eqn /. % // Flatten
r = eqn[[1]] /. Equal → Rule;
{eqn[[1]], eqn[[4]] /. r, eqn[[2]] /. r};
% // TableForm // FullSimplify
```

```
{-t v γ[v] + x γ[v], t γ[v] λ[v] - x γ[v] μ[v]} == {t u γ[u] + x γ[u], t γ[u] λ[u] + x γ[u] μ[u]}
```

```
γ[v] == γ[u]
-γ[v] μ[v] == γ[u] μ[u]
-v γ[v] == u γ[u]
γ[v] λ[v] == γ[u] λ[u]
```

```
InverseFunction::ifun : Inverse functions are being used. Values may be lost for multivalued inverses. More…

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General::stop : Further output of InverseFunction::ifun will be suppressed during this calculation. More…

Solve::incnst :
 Inconsistent or redundant transcendental equation. After reduction, the bad equation is (-u - v) γ[v] == 0. More…
```

```
{{u → -v}}
```

```
{γ[v] == γ[-v], -γ[v] μ[v] == γ[-v] μ[-v], -v γ[v] == -v γ[-v], γ[v] λ[v] == γ[-v] λ[-v]}
```

```
γ[-v] == γ[v]
γ[-v] (λ[-v] - λ[v]) == 0
γ[-v] (μ[-v] + μ[v]) == 0
```

HYPOTHESIS 3 : THE GROUP LAW

(a) Identity transformation

```
SolveAlways[{x, t} == {(-t w + x) γ[v], γ[v] (t λ[v] - x μ[v])}, {x, t}] /. v → w
```

```
{{w → 0, μ[w] → 0, λ[w] → 1, γ[w] → 1}}
```

(b) Inverse transformation

```
A = {x == (-t' w + x') γ[ww], t == γ[ww] (t' λ[ww] - x' μ[ww])};
B = Solve[{x', t'} == {(-t v + x) γ[vv], γ[vv] (t λ[vv] - x μ[vv])}, {x, t}] // Flatten;
SolveAlways[{A[[1, 2]] == B[[1, 2]], A[[2, 2]] == B[[2, 2]]}, {x', t'}] /. Rule → Equal;
Flatten[Table[Solve[%[[i]], {w, λ[ww], μ[ww], γ[ww]}], {i, 1, 3}], 1] /. {vv → v, ww → w} // MatrixForm
%[[2]] /. {Rule → Equal, λ[v] → 1}
%[[4]] /. w → -v /. γ[-v] → γ[v]
```

$$\begin{pmatrix} \mu[w] \to 0 & w \to -\frac{v}{\lambda[v]} & \lambda[w] \to \frac{1}{\lambda[v]} & \gamma[w] \to \frac{1}{\gamma[v]} \\ w \to -\frac{v}{\lambda[v]} & \mu[w] \to -\frac{\mu[v]}{\lambda[v]} & \lambda[w] \to \frac{1}{\lambda[v]} & \gamma[w] \to \frac{\lambda[v]}{\gamma[v]\,(\lambda[v]-v\,\mu[v])} \\ w \to 0 & \lambda[w] \to \frac{1}{\lambda[v]} & \mu[w] \to -\frac{\mu[v]}{\lambda[v]} & \gamma[w] \to \frac{1}{\gamma[v]} \end{pmatrix}$$

$$\left\{w == -v,\ \mu[w] == -\mu[v],\ \lambda[w] == 1,\ \gamma[w] == \frac{1}{\gamma[v]\,(1 - v\,\mu[v])}\right\}$$

$$\gamma[v] == \frac{1}{\gamma[v]\,(1 - v\,\mu[v])}$$

(λ[v]→1: see Luigi Masciovecchio "Lévy-Leblond ab omni naevo vindicatus")

(c) Composition law

```
Solve[{
   x1 == γ[v1] (x - v1 t), t1 == γ[v1] (t - μ[v1] x),
   x2 == γ[v2] (x1 - v2 t1), t2 == γ[v2] (t1 - μ[v2] x1)},
  {x2, t2}, {x1, t1}] // Flatten
V = -Coefficient[%[[1, 2]], t] / Coefficient[%[[1, 2]], x] // Simplify
Solve[Coefficient[%%[[1, 2]], x] == Coefficient[%%[[2, 2]], t], μ[v1]]
```

```
{x2 → (-t v1 + x) γ[v1] γ[v2] - v2 γ[v1] γ[v2] (t - x μ[v1]),
 t2 → γ[v1] γ[v2] (t - x μ[v1]) - (-t v1 + x) γ[v1] γ[v2] μ[v2]}
```

$$\frac{v1 + v2}{1 + v2\,\mu[v1]}$$

$$\left\{\left\{\mu[v1] \to \frac{v1\,\mu[v2]}{v2}\right\}\right\}$$

☹ `SolveAlways[μ[v1] / v1 == μ[v2] / v2, {v1, v2}, InverseFunctions → False]`

```
SolveAlways::tdep :
 The equations appear to involve the variables to be solved for in an essentially non-algebraic way. More…
```

$$\text{SolveAlways}\left[\frac{\mu[v1]}{v1} == \frac{\mu[v2]}{v2},\ \{v1, v2\},\ \text{InverseFunctions} \to \text{False}\right]$$

```
{V = V /. μ[v_] → α v,
 eq = γv == 1/(γv (1 - v μ[v])) /. μ[v] → α v}
Reduce[{eq, γv ≥ 0}, γv, Reals]
{"Non-causal:", Reduce[{% /. α → -κ^-2, κ > 0}, γv, Reals], V /. α → -κ^-2,
 "Galilean:", Reduce[{% /. α → 0}, γv, Reals], V /. α → 0,
 "Lorentz:", Reduce[{% /. α → c^-2, c > 0}, γv, Reals], V /. α → c^-2}
```

General::spell1 : Possible spelling error: new symbol name "γv" is similar to existing symbol "γ". More…

$$\left\{\frac{v1+v2}{1+v1\,v2\,\alpha},\ \gamma v == \frac{1}{(1-v^2\alpha)\,\gamma v}\right\}$$

$$\alpha \le 0 \;\&\&\; \gamma v == \sqrt{\frac{1}{1-v^2\alpha}} \;||\; \alpha > 0 \;\&\&\; -\sqrt{\frac{1}{\alpha}} < v < \sqrt{\frac{1}{\alpha}} \;\&\&\; \gamma v == \sqrt{\frac{1}{1-v^2\alpha}}$$

$$\left\{\text{Non-causal:},\ \kappa > 0 \;\&\&\; \gamma v == \sqrt{\frac{1}{1+\frac{v^2}{\kappa^2}}},\ \frac{v1+v2}{1-\frac{v1\,v2}{\kappa^2}},\ \text{Galilean:},\ \gamma v == 1,\right.$$
$$\left. v1+v2,\ \text{Lorentz:},\ (v \le 0 \;\&\&\; c > -v \;||\; v > 0 \;\&\&\; c > v) \;\&\&\; \gamma v == \sqrt{\frac{1}{1-\frac{v^2}{c^2}}},\ \frac{v1+v2}{1+\frac{v1\,v2}{c^2}}\right\}$$

HYPOTHESIS 4 : CAUSALITY

```
Xnc[x_, t_] := (1 + v^2 / κ^2)^(-1/2) (x - v t)
Tnc[x_, t_] := (1 + v^2 / κ^2)^(-1/2) (t + v x / κ^2)
Tnc[x2, t2] - Tnc[x1, t1] // FullSimplify;
{"Non-causal: Δt' = ", % /. (-t1 + t2) → Δt /. (-x1 + x2) → Δx}
```

General::spell1 : Possible spelling error: new symbol name "Δx" is similar to existing symbol "Δt". More…

$$\left\{\text{Non-causal: }\Delta t' = ,\ \frac{v\,\Delta x+\Delta t\,\kappa^2}{\sqrt{1+\frac{v^2}{\kappa^2}}\;\kappa^2}\right\}$$

```
Xl[x_, t_] := (1 - v^2 / c^2)^(-1/2) (x - v t)
Tl[x_, t_] := (1 - v^2 / c^2)^(-1/2) (t - v x / c^2)
Tl[x2, t2] - Tl[x1, t1] // FullSimplify;
Δt' = % /. (-t1 + t2) → Δt /. (x1 - x2) → -Δx;
Xl[x2, t2] - Xl[x1, t1] // Simplify;
Δx' = % /. v t1 - v t2 → -v Δt /. -(x1 - x2) → Δx;

{"Lorentz: Δt' = ", Δt', " Δx' = ", Δx'}
```

$$\left\{\text{Lorentz: }\Delta t' = ,\ \frac{c^2\,\Delta t - v\,\Delta x}{c^2\sqrt{1-\frac{v^2}{c^2}}},\ \ \Delta x' = ,\ \frac{-v\,\Delta t+\Delta x}{\sqrt{1-\frac{v^2}{c^2}}}\right\}$$

How can we derive the condition | Δx / Δt | < c in the Lorentz case with *Mathematica*?

```
☹ (c > 0 && -c < v < c) && (Δt > 0 ⇒ (c^2 Δt - v Δx > 0)) && (Δt < 0 ⇒ (c^2 Δt - v Δx < 0)) // LogicalExpand //
  FullSimplify
```

$$c > 0 \;\&\&\; c > v \;\&\&\; c+v > 0 \;\&\&\; \Delta t \ge 0 \;\&\&\; c^2\,\Delta t > v\,\Delta x \;||$$
$$c > 0 \;\&\&\; c > v \;\&\&\; c+v > 0 \;\&\&\; \Delta t == 0 \;||\; c > 0 \;\&\&\; c > v \;\&\&\; c+v > 0 \;\&\&\; \Delta t \le 0 \;\&\&\; c^2\,\Delta t < v\,\Delta x$$

```
☹ (c > 0 && -c < v < c && Δt ≠ 0) && c^2 > v Δx / Δt // LogicalExpand // FullSimplify
```

$$c > 0 \;\&\&\; c > v \;\&\&\; c+v > 0 \;\&\&\; \Delta t \ne 0 \;\&\&\; c^2 > \frac{v\,\Delta x}{\Delta t}$$

■ 1.4 Zweites Relativitätsargument.

From the preceding section we have:

```
(Λ[0, 0]  Λ[0, 1]) = γ ( 1   -v / c^2 );
(Λ[1, 0]  Λ[1, 1])     (-v      1     )
Lsk // MatrixForm
```

$$\begin{pmatrix} \gamma & -\frac{v\gamma}{c^2} & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & \Lambda[2,2] & 0 \\ 0 & 0 & 0 & \Lambda[2,2] \end{pmatrix}$$

Two frames obtained by the substitutions x→-x and z→-z in O and x'→-x' and z'→-z' in O' are connected by the same transformation matrix **Λ**, we have proved that v = -v' (reciprocity), we have also the continuity of **Λ**(v) and **Λ**(v=0) = **1** (identity). This implies:

```
xzInvertion = DiagonalMatrix[{1, -1, 1, -1}];
xzInvertion == Lsk.xzInvertion.Lsk /. γ → (1 - v^2 / c^2)^(-1/2) // Simplify
Solve[%]
Lsk = Lsk /. %[[2]];
```

```
{{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, -1}} ==
 {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, Λ[2, 2]^2, 0}, {0, 0, 0, -Λ[2, 2]^2}}
```

```
{{Λ[2, 2] → -1}, {Λ[2, 2] → 1}}
```

The Lorentz transformation for frames in standard configuration are finally:

```
MatrixForm /@ {Lsk, Lsk.Ereignis} // Simplify
```

$$\left\{ \begin{pmatrix} \gamma & -\frac{v\gamma}{c^2} & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \left(t - \frac{v x}{c^2}\right)\gamma \\ (-t v + x)\gamma \\ y \\ z \end{pmatrix} \right\}$$

■ 1.A Rechenbeispiele (examples of calculations).

□ *2. Allgemeine Lorentztransformation* (Lorentz transformation for frames in arbitrary configuration)

■ 2.1 Darstellungen

```
Clear[γ]
```

```
gr = γ → (1 - β^2 )^(-1/2);
grC = {γ → (1 - (vx^2 + vy^2 + vz^2) / c^2 )^(-1/2), v → Sqrt[vx^2 + vy^2 + vz^2]};
```

I. Spezielle Lorentztransformation. Koordinatensysteme in Standardkonfiguration. Relative Geschwindigkeit $\vec{v}$ = (v, 0, 0). Lorentz transformation for frames in standard configuration.

$$\Lambda slt = \begin{pmatrix} \gamma & -\frac{\beta\gamma}{c} & 0 & 0 \\ -\beta c \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \ Ereignis = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix};$$

```
Λslt /. gr /. β → v / c // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & -\frac{v}{c^2\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 \\ -\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

II. Allgemeine Lorentztransformation für Koordinatensysteme "ohne" relative Drehung.
Relative Geschwindigkeit $\vec{v}$ durch {ϕv, θv, v} oder durch $\vec{v}$ = (vx, vy, vz) gegeben.
Lorentz transformation for frames "without" relative rotation of the axis and moving with relative velocity $\vec{v}$.

$$Rv = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & Cos[\phi v] & 0 & Sin[\phi v] \\ 0 & 0 & 1 & 0 \\ 0 & -Sin[\phi v] & 0 & Cos[\phi v] \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & Cos[\theta v] & Sin[\theta v] & 0 \\ 0 & -Sin[\theta v] & Cos[\theta v] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

```
InvRv = Inverse[Rv] // Simplify;
MatrixForm /@ {Rv, InvRv}
```

General::spell1 : Possible spelling error: new symbol name "ϕv" is similar to existing symbol "γv". More…

General::spell : Possible spelling error: new symbol name "θv" is similar to existing symbols {γv, ϕv}. More…

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & Cos[\theta v]\,Cos[\phi v] & Cos[\phi v]\,Sin[\theta v] & Sin[\phi v] \\ 0 & -Sin[\theta v] & Cos[\theta v] & 0 \\ 0 & -Cos[\theta v]\,Sin[\phi v] & -Sin[\theta v]\,Sin[\phi v] & Cos[\phi v] \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & Cos[\theta v]\,Cos[\phi v] & -Sin[\theta v] & -Cos[\theta v]\,Sin[\phi v] \\ 0 & Cos[\phi v]\,Sin[\theta v] & Cos[\theta v] & -Sin[\theta v]\,Sin[\phi v] \\ 0 & Sin[\phi v] & 0 & Cos[\phi v] \end{pmatrix} \right\}$$

```
Λaltod = InvRv.Λslt.Rv // Simplify;
Λaltod[[2, 2]] = Λaltod[[2, 2]] /. {Sin[θv]^2 → (1 - Cos[θv]^2), Sin[ϕv]^2 → (1 - Cos[ϕv]^2)} // Simplify;
Λaltod[[3, 3]] = Λaltod[[3, 3]] /. {Cos[θv]^2 → (1 - Sin[θv]^2), Sin[ϕv]^2 → (1 - Cos[ϕv]^2)} // Simplify;
Λaltod[[4, 4]] = Λaltod[[4, 4]] /. {Cos[ϕv]^2 → (1 - Sin[ϕv]^2)} // Simplify;

AtCr = {Sin[θv] → vy / Sqrt[vx^2 + vy^2],
  Cos[θv] → vx / Sqrt[vx^2 + vy^2], Sin[ϕv] → vz / v, Cos[ϕv] → Sqrt[vx^2 + vy^2] / v, β → v / c};
ΛaltodC = Collect[Λaltod /. AtCr, vz^2 / v^2];

Λaltod // MatrixForm
ΛaltodC // MatrixForm
```

General::spell1 : Possible spelling error: new symbol name "ΛaltodC" is similar to existing symbol "Λaltod". More…

$$\begin{pmatrix} \gamma & -\frac{\beta\gamma\,Cos[\theta v]\,Cos[\phi v]}{c} & -\frac{\beta\gamma\,Cos[\phi v]\,Sin[\theta v]}{c} & -\frac{\beta \ldots \\ -c\,\beta\,\gamma\,Cos[\theta v]\,Cos[\phi v] & 1+(-1+\gamma)\,Cos[\theta v]^2\,Cos[\phi v]^2 & (-1+\gamma)\,Cos[\theta v]\,Cos[\phi v]^2\,Sin[\theta v] & (-1+\gamma)\,Cos[\theta \ldots \\ -c\,\beta\,\gamma\,Cos[\phi v]\,Sin[\theta v] & (-1+\gamma)\,Cos[\theta v]\,Cos[\phi v]^2\,Sin[\theta v] & 1+(-1+\gamma)\,Cos[\phi v]^2\,Sin[\theta v]^2 & (-1+\gamma)\,Cos[\phi \ldots \\ -c\,\beta\,\gamma\,Sin[\phi v] & (-1+\gamma)\,Cos[\theta v]\,Cos[\phi v]\,Sin[\phi v] & (-1+\gamma)\,Cos[\phi v]\,Sin[\theta v]\,Sin[\phi v] & 1+(-1 \ldots \end{pmatrix}$$

$$\begin{pmatrix} \gamma & -\frac{vx\,\gamma}{c^2} & -\frac{vy\,\gamma}{c^2} & -\frac{vz\,\gamma}{c^2} \\ -vx\,\gamma & 1+\frac{vx^2\,(-1+\gamma)}{v^2} & \frac{vx\,vy\,(-1+\gamma)}{v^2} & \frac{vx\,vz\,(-1+\gamma)}{v^2} \\ -vy\,\gamma & \frac{vx\,vy\,(-1+\gamma)}{v^2} & 1+\frac{vy^2\,(-1+\gamma)}{v^2} & \frac{vy\,vz\,(-1+\gamma)}{v^2} \\ -vz\,\gamma & \frac{vx\,vz\,(-1+\gamma)}{v^2} & \frac{vy\,vz\,(-1+\gamma)}{v^2} & 1+\frac{vz^2\,(-1+\gamma)}{v^2} \end{pmatrix}$$

III. Allgemeine Lorentztransformation für Koordinatensysteme "mit" relativer Drehung.
Relative Drehung durch {ψ, θ, ϕ} gegeben. Relative Geschwindigkeit $\vec{v}$ durch {ϕv, θv, v} oder durch $\vec{v}$ = (vx, vy, vz) gegeben. Lorentz transformation for frames "with" relative rotation of the axis given by the rotation matrix R and moving with relative velocity $\vec{v}$.

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{Cos}[\psi] & \mathrm{Sin}[\psi] & 0 \\ 0 & -\mathrm{Sin}[\psi] & \mathrm{Cos}[\psi] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{Cos}[\theta] & \mathrm{Sin}[\theta] \\ 0 & 0 & -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{Cos}[\phi] & \mathrm{Sin}[\phi] & 0 \\ 0 & -\mathrm{Sin}[\phi] & \mathrm{Cos}[\phi] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

```
Λaltmd = R.Λaltod // Simplify;
For[i = 1, i ≤ 4, i++,
 Print["\n", i, ". column of the tranformation matrix:\n", MatrixForm[Transpose[Λaltmd][[i]]] ]]
```

General::spell1 : Possible spelling error: new symbol name "Λaltmd" is similar to existing symbol "Λaltod". More…

1. column of the tranformation matrix:

```
γ
-c β γ (Sin[θ] Sin[ϕv] Sin[ψ] + Cos[ϕv] Sin[θv] (Cos[ψ] Sin[ϕ] + Cos[θ] Cos[ϕ] Sin[ψ]) + Cos[θv] Cos[ϕv] (Cos[ϕ] Cos[ψ] -
-c β γ (Cos[θ] Cos[ϕv] Cos[ψ] Sin[θv - ϕ] + Cos[ψ] Sin[θ] Sin[ϕv] - Cos[θv - ϕ] Cos[ϕv] Sin[ψ])
c β γ (Cos[ϕ] Cos[ϕv] Sin[θ] Sin[θv] - Cos[θv] Cos[ϕv] Sin[θ] Sin[ϕ] - Cos[θ] Sin[ϕv])
```

2. column of the tranformation matrix:

```
-(β γ Cos[θv] Cos[ϕv])/c
(-1 + γ) Cos[θv] Cos[ϕv] Sin[θ] Sin[ϕv] Sin[ψ] + (-1 + γ) Cos[θv] Cos[ϕv]^2 Sin[θv] (Cos[ψ] Sin[ϕ] + Cos[θ] Cos[ϕ] Sin[ψ]
(-1 + γ) Cos[θv] Cos[ϕv] Cos[ψ] Sin[θ] Sin[ϕv] + (1 + (-1 + γ) Cos[θv]^2 Cos[ϕv]^2) (-Cos[θ] Cos[ψ] Sin[ϕ] - Cos[ϕ] Sin[ψ])
Sin[θ] Sin[ϕ] + (-1 + γ) Cos[θv]^2 Cos[ϕv]^2 Sin[θ] Sin[ϕ] - (-1 + γ) Cos[θv] Cos[ϕv] (Cos[ϕ
```

3. column of the tranformation matrix:

```
-(β γ Cos[ϕv] Sin[θv])/c
(-1 + γ) Cos[ϕv] Sin[θ] Sin[θv] Sin[ϕv] Sin[ψ] + (1 + (-1 + γ) Cos[ϕv]^2 Sin[θv]^2) (Cos[ψ] Sin[ϕ] + Cos[θ] Cos[ϕ] Sin[ψ]) +
(-1 + γ) Cos[ϕv] Cos[ψ] Sin[θ] Sin[θv] Sin[ϕv] - (-1 + γ) Cos[θv] Cos[ϕv]^2 Sin[θv] (Cos[θ] Cos[ψ] Sin[ϕ] + Cos[ϕ] Sin[ψ])
-Cos[ϕ] Sin[θ] (1 + (-1 + γ) Cos[ϕv]^2 Sin[θv]^2) + (-1 + γ) Cos[ϕv] Sin[θv] (Cos[θv] Cos
```

4. column of the tranformation matrix:

```
-(β γ Sin[ϕv])/c
Sin[θ] (1 + (-1 + γ) Sin[ϕv]^2) Sin[ψ] + (-1 + γ) Cos[ϕv] Sin[θv] Sin[ϕv] (Cos[ψ] Sin[ϕ] + Cos[θ] Cos[ϕ] Sin[ψ]) + (-1 + γ) C
Cos[ψ] (Sin[θ] (1 + (-1 + γ) Sin[ϕv]^2) + 1/2 (-1 + γ) Cos[θ] Sin[θv - ϕ] Sin[2 ϕv]) - 1/2 (-1 +
Cos[θ] (1 + (-1 + γ) Sin[ϕv]^2) - 1/2 (-1 + γ) Sin[θ] Sin[θv - ϕ] Sin
```

```
ΛaltmdC = R.ΛaltodC // Simplify;
For[i = 1, i ≤ 4, i++,
 Print["\n", i, ". column of the tranformation matrix:\n", MatrixForm[Transpose[ΛaltmdC][[i]]] ]]
```

General::spell : Possible spelling error: new symbol name "ΛaltmdC" is similar to existing symbols {Λaltmd, ΛaltodC}. More…

1. column of the tranformation matrix:

$$\begin{pmatrix} \gamma \\ -\gamma\,(\mathrm{vy}\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] + (\mathrm{vz}\,\mathrm{Sin}[\theta] - \mathrm{vx}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi])\,\mathrm{Sin}[\psi] + \mathrm{Cos}[\phi]\,(\mathrm{vx}\,\mathrm{Cos}[\psi] + \mathrm{vy}\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\psi])) \\ \gamma\,(-\mathrm{vz}\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] - \mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi]\,(\mathrm{vy}\,\mathrm{Cos}[\phi] - \mathrm{vx}\,\mathrm{Sin}[\phi]) + (\mathrm{vx}\,\mathrm{Cos}[\phi] + \mathrm{vy}\,\mathrm{Sin}[\phi])\,\mathrm{Sin}[\psi]) \\ -\gamma\,(\mathrm{vz}\,\mathrm{Cos}[\theta] + \mathrm{Sin}[\theta]\,(-\mathrm{vy}\,\mathrm{Cos}[\phi] + \mathrm{vx}\,\mathrm{Sin}[\phi])) \end{pmatrix}$$

2. column of the tranformation matrix:

$$\begin{pmatrix} -\frac{\mathrm{vx}\,\gamma}{c^2} \\ \frac{\mathrm{vx}\,\mathrm{vz}\,(-1+\gamma)\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi]}{v^2} + \frac{\mathrm{vx}\,\mathrm{vy}\,(-1+\gamma)\,(\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi]+\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi])}{v^2} + \left(1 + \frac{\mathrm{vx}^2\,(-1+\gamma)}{v^2}\right)(\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] - \mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi]) \\ \frac{\mathrm{vx}\,\mathrm{vz}\,(-1+\gamma)\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]}{v^2} + \left(1 + \frac{\mathrm{vx}^2\,(-1+\gamma)}{v^2}\right)(-\mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] - \mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi]) + \frac{\mathrm{vx}\,\mathrm{vy}\,(-1+\gamma)\,(\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi]-\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi])}{v^2} \\ \frac{\mathrm{vx}\,\mathrm{vz}\,(-1+\gamma)\,\mathrm{Cos}[\theta]+\mathrm{Sin}[\theta]\,(-\mathrm{vx}\,\mathrm{vy}\,(-1+\gamma)\,\mathrm{Cos}[\phi]+(v^2+\mathrm{vx}^2\,(-1+\gamma))\,\mathrm{Sin}[\phi])}{v^2} \end{pmatrix}$$

3. column of the tranformation matrix:

$$\begin{pmatrix} -\frac{\mathrm{vy}\,\gamma}{c^2} \\ \frac{\mathrm{vy}\,\mathrm{vz}\,(-1+\gamma)\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi]}{v^2} + \left(1 + \frac{\mathrm{vy}^2\,(-1+\gamma)}{v^2}\right)(\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] + \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi]) + \frac{\mathrm{vx}\,\mathrm{vy}\,(-1+\gamma)\,(\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi]-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi])}{v^2} \\ \frac{\mathrm{vy}\,\mathrm{vz}\,(-1+\gamma)\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]}{v^2} - \frac{\mathrm{vx}\,\mathrm{vy}\,(-1+\gamma)\,(\mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi]+\mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi])}{v^2} + \left(1 + \frac{\mathrm{vy}^2\,(-1+\gamma)}{v^2}\right)(\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] - \mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi]) \\ \frac{\mathrm{vy}\,\mathrm{vz}\,(-1+\gamma)\,\mathrm{Cos}[\theta]-\mathrm{Sin}[\theta]\,((v^2+\mathrm{vy}^2\,(-1+\gamma))\,\mathrm{Cos}[\phi]-\mathrm{vx}\,\mathrm{vy}\,(-1+\gamma)\,\mathrm{Sin}[\phi])}{v^2} \end{pmatrix}$$

4. column of the tranformation matrix:

$$\begin{pmatrix} -\frac{vz\,\gamma}{c^2} \\ \frac{vy\,vz\,(-1+\gamma)\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi]+((v^2+vz^2\,(-1+\gamma))\,\mathrm{Sin}[\theta]-vx\,vz\,(-1+\gamma)\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi])\,\mathrm{Sin}[\psi]+vz\,(-1+\gamma)\,\mathrm{Cos}[\phi]\,(vx\,\mathrm{Cos}[\psi]+vy\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\psi])}{v^2} \\ \frac{(v^2+vz^2\,(-1+\gamma))\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]+vz\,(-1+\gamma)\,\mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi]\,(vy\,\mathrm{Cos}[\phi]-vx\,\mathrm{Sin}[\phi])-vz\,(-1+\gamma)\,(vx\,\mathrm{Cos}[\phi]+vy\,\mathrm{Sin}[\phi])\,\mathrm{Sin}[\psi]}{v^2} \\ \frac{(v^2+vz^2\,(-1+\gamma))\,\mathrm{Cos}[\theta]-vz\,(-1+\gamma)\,\mathrm{Sin}[\theta]\,(vy\,\mathrm{Cos}[\phi]-vx\,\mathrm{Sin}[\phi])}{v^2} \end{pmatrix}$$

■ 2.2 Beweis der Isometrie der Lorentztransformation im Minkowsi-Raum. Fälle I, II und III. Proof of the isometry of the transformation formulas in the Minkowski space. Cases I, II and III.

```
STI := (c EP[[1]])^2 - EP[[2]]^2 - EP[[3]]^2 - EP[[4]]^2;
Print["Case I."];
EP = Λslt.Ereignis; STI /. gr // Simplify
Print["Case II."];
EP = Λaltod.Ereignis; STI /. gr // Simplify
EP = ΛaltodC.Ereignis; STI /. grC // Simplify
Print["Case III."];
EP = Λaltmd.Ereignis; Timing[ STI /. gr // Simplify]
EP = ΛaltmdC.Ereignis; Timing[STI /. grC // Simplify]
```

Case I.

$\{c^2 t^2 - x^2 - y^2 - z^2\}$

Case II.

$\{c^2 t^2 - x^2 - y^2 - z^2\}$

$\{c^2 t^2 - x^2 - y^2 - z^2\}$

Case III.

$\{210.14\ \text{Second},\ \{c^2 t^2 - x^2 - y^2 - z^2\}\}$

$\{301.93\ \text{Second},\ \{c^2 t^2 - x^2 - y^2 - z^2\}\}$

```
(* Comparazione Timing desktop-laptop *)
(174.5 + 174.39 + 172.69) / (43.603 + 43.853 + 43.573)
```

3.98065

■ 2.A Vergleich meiner Ergebnisse mit denen vom *TGeneralRelativity1`GeneralRelativity` Mathematica Tensorial subpackage* von David Park. Comparison of results.

```
a = Λaltod /. gr // MatrixForm;
{a /. {β -> β, ϕv → 0, θv -> 0},
 a /. {β -> β, ϕv → Pi / 2, θv -> 0},
 a /. {β → 0.24, ϕv → 3, θv → 1964}}
```

$$\left\{\begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta}{c\sqrt{1-\beta^2}} & 0 & 0 \\ -\frac{c\,\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & -\frac{\beta}{c\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{c\,\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix}, \begin{pmatrix} 1.03011 & -\frac{0.214249}{c} & -\frac{0.118325}{c} & -\frac{0.0348885}{c} \\ -0.214249\,c & 1.02261 & 0.0124875 & 0.00368198 \\ -0.118325\,c & 0.0124875 & 1.0069 & 0.00203348 \\ -0.0348885\,c & 0.00368198 & 0.00203348 & 1.0006 \end{pmatrix}\right\}$$

```
⊗ Needs["TGeneralRelativity1`GeneralRelativity`"]
```

```
$PrePrint =.
DeclareBaseIndices[{0, 1, 2, 3}]
DeclareIndexFlavor[{red, Red}]
DefineTensorShortcuts[{ΛP, 2}]
```

```
SetLorentzBoost[ΛP, red, Identity][β, ϕ, θ]
ΛPud[red@α, β] == (ΛPud[red@α, β] // ToArrayValues[]);
Last[%] // MatrixForm
SetLorentzBoost[ΛP, red, Identity][β, 0, 0]
ΛPud[red@α, β] == (ΛPud[red@α, β] // ToArrayValues[]);
a = Last[%] // MatrixForm;
SetLorentzBoost[ΛP, red, Identity][β, π / 2, 0]
ΛPud[red@α, β] == (ΛPud[red@α, β] // ToArrayValues[]);
b = Last[%] // MatrixForm;
SetLorentzBoost[ΛP, red, Identity][0.24, 3, 1964]
ΛPud[red@α, β] == (ΛPud[red@α, β] // ToArrayValues[]);
c = Last[%] // MatrixForm; {a, b, c}
```

$$\begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta\,\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]}{\sqrt{1-\beta^2}} & -\frac{\beta\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]}{\sqrt{1-\beta^2}} & -\frac{\beta\,\mathrm{Sin}[\phi}{\sqrt{1-\beta^2}} \\ -\frac{\beta\,\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]}{\sqrt{1-\beta^2}} & 1+\left(-1+\frac{1}{\sqrt{1-\beta^2}}\right)\mathrm{Cos}[\theta]^2\,\mathrm{Cos}[\phi]^2 & \left(-1+\frac{1}{\sqrt{1-\beta^2}}\right)\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]^2\,\mathrm{Sin}[\theta] & \left(-1+\frac{1}{\sqrt{1-\beta^2}}\right)\mathrm{Cos}[\theta] \\ -\frac{\beta\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]}{\sqrt{1-\beta^2}} & \left(-1+\frac{1}{\sqrt{1-\beta^2}}\right)\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]^2\,\mathrm{Sin}[\theta] & 1+\left(-1+\frac{1}{\sqrt{1-\beta^2}}\right)\mathrm{Cos}[\phi]^2\,\mathrm{Sin}[\theta]^2 & \left(-1+\frac{1}{\sqrt{1-\beta^2}}\right)\mathrm{Cos}[\phi] \\ -\frac{\beta\,\mathrm{Sin}[\phi]}{\sqrt{1-\beta^2}} & \left(-1+\frac{1}{\sqrt{1-\beta^2}}\right)\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\phi] & \left(-1+\frac{1}{\sqrt{1-\beta^2}}\right)\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] & 1+\left(-1+\frac{1}{\sqrt{1-\beta^2}}\right. \end{pmatrix}$$

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta}{\sqrt{1-\beta^2}} & 0 & 0 \\ -\frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 & -\frac{\beta}{\sqrt{1-\beta^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix}, \begin{pmatrix} 1.03011 & -0.214249 & -0.118325 & -0.0348885 \\ -0.214249 & 1.02261 & 0.0124875 & 0.00368198 \\ -0.118325 & 0.0124875 & 1.0069 & 0.00203348 \\ -0.0348885 & 0.00368198 & 0.00203348 & 1.0006 \end{pmatrix} \right\}$$

ANHANG

Definitions of the SI base units.

The *second* is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.

The *meter* is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.

```
<< Miscellaneous`PhysicalConstants`
{SpeedOfLight}
```

$$\left\{\frac{299792458\,\mathrm{Meter}}{\mathrm{Second}}\right\}$$

"That's all Folks!"

* * *