General relativity

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```
Print["Revision ", IntegerPart[Date[]]]
```

```
Revision {2012, 9, 24, 19, 18, 56}
```

A) INTRODUCTION

Dear Colleagues,

This is my personal *Mathematica* notebook on Albert Einstein's genial **general theory of relativity**. This document wasn't originally intended for publication, but a few formulas and tricks are maybe of interest to you, so here they are. The code seems to work well and I added some comments to make it more understandable. This is not an introduction to this field, so use it at your own risk!

The main point about this work is to show how to do the typical mathematics of GR easily and rigorously with *Mathematica*. In addition, I "streamlined" a little bit the derivation of some classical results (perihelion advance, bending of light etc.).

As main textbook I have chosen the excellent and brilliantly instructive "A short course in general relativity" by James Foster and J.David Nightingale. *Mathematica* together with the packages Tensorial and GeneralRelativity have been used by David Park to do all the derivations, examples and exercises of this textbook. Most of the present notebook is actually a rewrite of Park's very fine original work.

Once again, the combination of a good textbook and *Mathematica* provides a fun, easy and mathematical rigorous learning environment that stimulates greatly understanding and own experiments with the formulas. Don't miss it!

* * *

General relativity is a metric theory of gravitation. At its core are Einstein's field equations, which describe the relation between the geometry of a four-dimensional, pseudo-Riemannian manifold representing spacetime, and the energy-momentum contained in that spacetime. First published by Albert Einstein in 1915 as a tensor equation, the Einstein's field equations equate spacetime curvature (expressed by the Einstein tensor) with the energy and momentum within that spacetime (expressed by the energy-momentum-stress tensor). General relativity's predictions have been confirmed in all observations and experiments to date. Although general relativity is not the only relativistic theory of gravity, it is the simplest theory that is consistent with experimental data. (*Wikipedia*, 2011)

General relativity is a geometric theory and incorporates special relativity in the sense that locally the spacetime of the general theory is like that of the special theory. So it's important for the sake of conceptual cleanness to derive in your course first special relativity from the basic geometrical spacetime symmetries without using the postulate of constant speed of light or any other "unneeded physics" (see for example Jean-Marc Lévy-Leblond, "One more derivation of the Lorentz transformation", American Journal of Physics **44**, 271-277 (1976); visit http://o.castera.free.fr for more information).

Valuable web resources on general relativity:

• David Park, *Mathematica* notebooks (2005) based on "A short course in general relativity" (Foster/Nightingale)
• See books.google.com or the Springer editor web site for a preview of the above-mentioned textbook.
• Florian Schrack "Gravitation - Theorien, Effekte und Simulation am Computer" (2002)
• Gerard 't Hooft "Introduction to General Relativity" (2007)
• Matt Visser "Math 464: Notes on Differential Geometry" (2009)
• Matt Visser "Math 465: Notes on General Relativity and Cosmology" (2009)
• Norbert Dragon "Geometrie der Relativitätstheorie" (2011)
• Sean Carroll "Lecture Notes on General Relativity" (1997)
• Tom Marsh "Notes for PX436, General Relativity" (2009)
• Clifford M. Will "The Confrontation between General Relativity and Experiment", Living Rev. Relativity, 9, (2006)
• Neil Ashby "Relativity in the Global Positioning System", Living Rev. Relativity, 6, (2003)
• Wikipedia: "General relativity", "Allgemeine Relativitätstheorie" and links
• General relativity video courses (Charles Bailyn, Alexander Maloney, Lenny Susskind)

Note:

• ***Mathematica*** by Wolfram Research is a (fabulous) computer algebra system.
• A **notebook** is an interactive *Mathematica* document (extension .nb).
• **Tensorial 3.0** (R. Cabrera, D. Park, J.-F. Gouyet, August 2005) is a general-purpose tensor calculus package for *Mathematica* Version 4.1 or later.
• **TGeneralRelativity1`GeneralRelativity`** (D. Park, 29 January 2005) is a subpackage for the Tensorial package that adds routines useful in special and general relativity. (This also automatically loads the regular Tensorial package.)

```
Print["This system is:"]
{"ProductIDName", "ProductVersion"} /. $ProductInformation
ReadList["!ver", String][[2]]
{$MachineType, $ProcessorType, $ByteOrdering, $SystemCharacterEncoding}
```

```
This system is:

{Mathematica, 5.2 for Microsoft Windows (June 20, 2005)}

Windows 98 [versione 4.10.1998]

{PC, x86, -1, WindowsANSI}
```

B) HELP

(Extracted from the Tensorial package help.)

■ **{x,δ,g,Γ}** are the standard set of tensor labels used in all Tensorial derivative routines. They tell the routines which labels will be considered to represent the coordinates x, Kronecker δ, metric tensor g and Christoffel symbol Γ.

■ **DeclareBaseIndices[{index..}]** declares the base indices for the underlying linear space.

■ **DeclareIndexFlavor[{flavorname,flavorform}...]** will add the index flavors to the IndexFlavors list and establish the Format for displaying indices with the given flavor name.

■ **ToArrayValues[baseindices][expr]** will convert the expression to a vector, matrix or array by expansion and substitution of any stored values.

■ **EvaluateDotProducts[e,g,metricsimplify:True][expr]** expands Dot products of vectors expressed in a given basis e using the metric tensor g. Metric simplification is performed if the default argument metricsimplify is True.

■ **LinearBreakout[f1,f2,...][v1,v2,...][expr]** will break out the linear terms of any expressions within expr that have heads matching the patterns fi over variables matching the patterns vj.

■ **SetMetricValues[g,metricmatrix,flavor:Identity]** creates value definitions for the up and down forms of the metric tensor using the label g and a metric matrix.

■ **CoordinateToTensors[{r,θ,ϕ...},coord:x,flavor:Identity][expr]** will convert the coordinate symbols in the expression to the corresponding indexed tensors. The optional arguments coord and flavor give the coordinate label and index flavor to use. Their default values are x and plain.

■ **SetChristoffelValueRules[xu[i,metricmatrix,Γ,simplification:Identity]** calculates and stores substitution rules for the Christoffel values of Γudd[i,j,k] and Γddd[i,j,k] from the values of metricmatrix and the xu[i] vector pattern.

■ **SelectedTensorRules[label,pattern]** will select the rules for label whose right hand sides are nonzero and whose left hand sides match the pattern.

■ **SimplifyTensorSum[expr]** will check that all terms in a tensor sum have valid indices,that the free indices are the same in all terms,and will simplify the sum by matching dummy indices in all terms that have the same index structure.

■ **ExpandCovariantD[{x,δ,g,Γ},a][expr]** will expand first order covariant derivatives of tensors using x as the label for the coordinates, δ as the label for the Kronecker, g as the label for the metric tensor and Γ as the label for Christoffel symbols. The introduced dummy index will be a.

■ **MapLevelParts[function,{topposition,levelpositions}][expr]** will map the function onto the selected level positions in an expression. The function is applied to them as a group and they are replaced with a single new expression. Other parts not specified on the list are left unchanged.

C) PHYSICAL CONSTANTS

Some physical constants as given by *Mathematica*.

```
Print["Miscellaneous`PhysicalConstants`:"]
<< Miscellaneous`PhysicalConstants`
<< Miscellaneous`Units`
{SpeedOfLight, GravitationalConstant, CosmicBackgroundTemperature, HubbleConstant}
{HubbleConstant^-1, AgeOfUniverse / HubbleConstant^-1, Convert[AgeOfUniverse, Year]}
{"Earth:", EarthMass, EarthRadius, "Sun:",
 Convert[SolarSchwarzschildRadius SpeedOfLight^2 / (2 GravitationalConstant), Kilogram],
 SolarRadius, SolarSchwarzschildRadius}
```

Miscellaneous`PhysicalConstants`:

$\left\{\frac{299792458\ \text{Meter}}{\text{Second}}, \frac{6.673\times10^{-11}\ \text{Meter}^2\ \text{Newton}}{\text{Kilogram}^2}, 2.726\ \text{Kelvin}, \frac{3.2\times10^{-18}}{\text{Second}}\right\}$

$\{3.125\times10^{17}\ \text{Second}, 1.504, 1.49036\times10^{10}\ \text{Year}\}$

$\{\text{Earth:}, 5.9742\times10^{24}\ \text{Kilogram}, 6378140\ \text{Meter}, \text{Sun:}, 1.9888\times10^{30}\ \text{Kilogram}, 6.9599\times10^{8}\ \text{Meter}, 2953.25\ \text{Meter}\}$

```
Print["Gravitational constant\nG = ",
 Convert[GravitationalConstant, Kilogram^-1 Meter^3 Second^-2], " = ",
 Convert[GravitationalConstant, Gram^-1 Centimeter^3 Second^-2]]
Convert[8 π GravitationalConstant / SpeedOfLight^2, Meter / Kilogram];
Print[
 "Einsteinsche Gravitationskonstante in Sexl/Urbantke S.69\nκ = 8 π G c^-2 = ", %, " = ",
 Convert[%, Gram^-1 Centimeter]]
Print["coupling constant in Foster/Nightingale p.113\nκ = - 8 π G c^-4 = ",
 - %% / SpeedOfLight^2]
```

Gravitational constant

$$G = \frac{6.673\times10^{-11}\ \text{Meter}^3}{\text{Kilogram Second}^2} = \frac{6.673\times10^{-8}\ \text{Centimeter}^3}{\text{Gram Second}^2}$$

Einsteinsche Gravitationskonstante in Sexl/Urbantke S.69

$$\kappa = 8\ \pi\ G\ c^{-2} = \frac{1.86603\times10^{-26}\ \text{Meter}}{\text{Kilogram}} = \frac{1.86603\times10^{-27}\ \text{Centimeter}}{\text{Gram}}$$

coupling constant in Foster/Nightingale p.113

$$\kappa = -\ 8\ \pi\ G\ c^{-4} = -\frac{2.07624\times10^{-43}\ \text{Second}^2}{\text{Kilogram Meter}}$$

I will use the CODATA 2010 values. (See http://physics.nist.gov/ for updates.)

```
Print["CODATA 2010: G = 6.673 84(80)×10^-11 m^3 kg^-1 s^-2"]
Print["κ = - 8 π G c^-4 = ", NumberForm[-8 π 6.67384 × 10^-11 / (299792458^4), 7], " m^-1 kg^-1 s^2"]
```

CODATA 2010: $G = 6.673\ 84(80)\times10^{-11}\ m^3\ kg^{-1}\ s^{-2}$

$\kappa = -\ 8\ \pi\ G\ c^{-4} = -2.076504\times10^{-43}\ m^{-1}\ kg^{-1}\ s^2$

D) OWN (?) CONSIDERATIONS

Special relativity teaches us how spacetime dictates the behavior of matter-energy and general relativity teaches us how matter-energy influences the behavior of spacetime. We could say that this two entities, spacetime and matter-energy, are in some kind of interaction. Starting from a heuristic principle that states that entities who can interact can not be *completely* different "in essence", we could tentatively postulate a symmetry between spacetime and matter-energy, implying the possibility of a transformation of spacetime into matter-energy and vice-versa. So it's maybe sensible to ask:

• How much spacetime can we get from 1 joule of matter-energy or vice-versa? What is the conversion factor λ between (geometrized) spacetime and matter-energy ($1\ m^4 \triangleq \lambda \cdot 1\ J$)?

• What are the observable signatures of spacetime ⇄ matter-energy transformations?

• How "expands" newly created spacetime in some finite region into the rest of the universe? How works the local "collapse" of the universe caused by the destruction of a finite piece of spacetime?

• How works the spacetime - matter-energy - transformation at a fundamental level?